

## NOTES ON GEOMETRIC LANGLANDS: CRYSTALS AND D-MODULES

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ABSTRACT. The goal of this paper is to develop the notion of crystal in the context of derived algebraic geometry, and to connect crystals to more classical objects such as D-modules.

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## INTRODUCTION

### 0.1. Flat connections, D-modules and crystals.

0.1.1. Let  $M$  be a smooth manifold with a vector bundle  $V$ . Recall that a flat connection on  $V$  is a map

$$\nabla : V \rightarrow V \otimes \Omega_M^1$$

satisfying the Leibniz rule, and such that the curvature  $[\nabla, \nabla] = 0$ . Dualizing the connection map, we obtain a map

$$T_M \otimes V \rightarrow V.$$

The flatness of the connection implies that this makes  $V$  into a module over the Lie algebra of vector fields. Equivalently, we obtain that  $V$  is module over the algebra  $\mathrm{Diff}_M$  of differential operators on  $M$ .

This notion generalizes immediately to smooth algebraic varieties in characteristic zero. On such a variety a D-module is defined as a module over the sheaf of differential operators which is quasi-coherent as an  $\mathcal{O}$ -module. The notion of D-module on an algebraic variety thus generalizes the notion of vector bundle with a flat connection, and encodes the data of a system of linear differential equations with polynomial coefficients. The study of D-modules on smooth algebraic varieties is a very rich theory, with applications to numerous fields such as representation theory. Many of the ideas from the differential geometry of vector bundles with a flat connection carry over to this setting.

However, the above approach to D-modules presents a number of difficulties. For example, one needs to consider sheaves with a flat connection on singular schemes in addition to smooth ones. While the algebra of differential operators is well-defined on a singular variety, the category of modules over it is not *the* category that we are interested in (e.g., the algebra in question is not in general Noetherian). In another direction, even for a smooth algebraic variety, it is not clear how to define connections on objects that are not linear, e.g., sheaves of categories.

0.1.2. *Parallel transport.* The idea of a better definition comes from another interpretation of the notion of flat connection on a vector bundle in the context of differential geometry, namely, that of parallel transport:

Given a vector bundle with a flat connection  $V$  on a smooth manifold  $M$ , and a path  $\gamma : [0, 1] \rightarrow M$ , we obtain an isomorphism

$$\Pi_\gamma : V_{\gamma(0)} \simeq V_{\gamma(1)}$$

of the fibers of  $V$  at the endpoints, which only depends on the homotopy class of the path. We can rephrase this construction as follows. Let  $B \subset M$  be a small ball inside  $M$ . Since the parallel transport isomorphism only depends on the homotopy class of the path, and  $B$  is contractible, we obtain a coherent identification of fibers of  $V$

$$V_x \simeq V_y$$

for points  $x, y \in B$ . So, roughly, the data of a connection gives an identification of fibers at “nearby” points of the manifold.

Building on this idea, Grothendieck [Gr] gave a purely algebraic analogue of the notion of parallel transport, using the theory of schemes (rather than just varieties) in an essential way: he introduced the relation of infinitesimal closeness for  $R$ -points of a scheme  $X$ . Namely, two  $R$ -points  $x, y : \mathrm{Spec}(R) \rightarrow X$  are infinitesimally close if the restrictions to  $\mathrm{Spec}({}^{red}R)$  agree, where  ${}^{red}R$  is the quotient of  $R$  by its nilradical.

A *crystal* on  $X$  is by definition a quasi-coherent sheaf on  $X$  which is equivariant with respect to the relation of infinitesimal closeness. More precisely, a crystal on  $X$  is a quasi-coherent sheaf  $\mathcal{F}$  with the additional data of isomorphisms

$$x^*(\mathcal{F}) \simeq y^*(\mathcal{F})$$

for any two infinitesimally close points  $x, y : \mathrm{Spec}(R) \rightarrow X$  satisfying a cocycle condition.

Grothendieck showed that on a smooth algebraic variety, the abelian category of crystals is equivalent to that of left modules over the ring of differential operators. In this way, crystals give a more fundamental definition of sheaves with a flat connection.

A salient feature of the category of crystals is that Kashiwara’s lemma is built into its definition: for a closed embedding of schemes  $i : Z \rightarrow X$ , the category of crystals on  $Z$  is equivalent to the category of crystals on  $X$ , which are set-theoretically supported on  $Z$ . This observation allows us to reduce the study of crystals on schemes to the case of smooth schemes, by (locally) embedding a given scheme into a smooth one.

0.1.3. In this paper, we develop the theory of crystals in the context of derived algebraic geometry, where instead of ordinary rings one considers derived rings, i.e.,  $E_\infty$  ring spectra. Since we are working over a field  $k$  of characteristic zero, we shall use connective commutative DG  $k$ -algebras as our model of derived rings (accordingly, we shall use the term “DG scheme” rather than “derived scheme”). The key idea is that one should regard higher homotopy groups of a derived ring as a generalization of nilpotent elements.

Thus, following Simpson [Si], for a DG scheme  $X$ , we define its de Rham prestack  $X_{\mathrm{dR}}$  to be the functor

$$X_{\mathrm{dR}} : R \mapsto X^{(red, cl)}(R)$$

on the category of derived rings  $R$ , where

$${}^{red, cl}R := {}^{red}(\pi_0(R))$$

is the reduced ring corresponding to the underlying classical ring of  $R$ . I.e.,  $X_{\text{dR}}$  is a *prestack* in the terminology of [GL:Stacks].

We define crystals on  $X$  as quasi-coherent sheaves on the prestack  $X_{\text{dR}}$ . See, [Lu1, Sect. 2] for the theory of quasi-coherent sheaves in prestacks, or [GL:QCoh, Sect. 1.1] for a brief review.

The above definition does not coincide with one of Grothendieck mentioned earlier: the latter specifies a map  $\text{Spec}(R) \rightarrow X$  up to an equivalence relation, and the former only a map  $\text{Spec}^{(\text{red}, \text{cl})}(R) \rightarrow X$ . However, we will show that for  $X$  which is *eventually coconnective*, i.e., if its structure ring has only finitely many non-zero homotopy groups, the two definitions of a crystal are equivalent.<sup>1</sup>

0.1.4. Even though the category of crystals is equivalent to that of D-modules, it offers a more flexible framework in which to develop the theory. The definition immediately extends to non-smooth schemes, and the corresponding category is well-behaved (for instance, the category of crystals on any scheme is Noetherian).

Let  $f : X \rightarrow Y$  be a map of DG schemes. We will construct the natural pullback functor

$$f^\dagger : \text{Crys}(Y) \rightarrow \text{Crys}(X).$$

In fact, we shall extend the assignment  $X \mapsto \text{Crys}(X)$  to a functor from the category  $\text{DGSch}^{op}$  to that of stable  $\infty$ -categories. The latter will enable us to define crystals not just on DG schemes, but on arbitrary prestacks.

Furthermore, the notion of crystal immediately extends to a non-linear and categorified setting. Namely, we can just as well define a crystal of schemes or a crystal of categories over  $X$ .

## 0.2. Left crystals vs. right crystals.

0.2.1. Recall that on a smooth algebraic variety  $X$ , in addition to usual (i.e., left) D-modules, one can also consider the category of right D-modules. The two categories are equivalent: the corresponding functor is given by tensoring with the dualizing line bundle  $\omega_X$  over the ring of functions. However, this equivalence does not preserve the forgetful functor to quasi-coherent sheaves. For this reason, we can consider an abstract category of D-modules, with two different realization functors to quasi-coherent sheaves. In the left realization, the D-module pullback functor becomes the  $*$ -pullback functor on quasi-coherent sheaves, and in the right realization, it becomes the  $!$ -pullback functor.

It turns out that the “right” realization has several advantages over the “left” one. Perhaps the main advantage is that the “right” realization endows the category of D-modules with a t-structure with very favorable functorial properties. In particular, this t-structure becomes the perverse t-structure under the Riemann-Hilbert correspondence.

0.2.2. One can then ask whether there are also “left” and “right” crystals on arbitrary DG schemes. It turns out that indeed both categories are defined very generally.

Left crystals are what we defined in Sect. 0.1.3. However, in order to define right crystals, we need to replace the usual category of quasi-coherent sheaves by its renormalized version, the category of ind-coherent sheaves introduced in [GL:IndCoh].

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<sup>1</sup>When  $X$  is not eventually coconnective, the two notions are different, and the correct one is the one via  $X_{\text{dR}}$ .

The category  $\mathrm{IndCoh}(X)$  is well-behaved for (derived) schemes that are (almost) locally of finite type, so right crystals will only be defined on DG schemes, and subsequently, on prestacks with this property.

Let us recall from [GL:IndCoh, Sect. 5] that for a map  $f : X \rightarrow Y$  between DG schemes, we have the  $!$ -pullback functor

$$f^! : \mathrm{IndCoh}(Y) \rightarrow \mathrm{IndCoh}(X).$$

The assignment  $X \mapsto \mathrm{IndCoh}(X)$  is a functor from the category  $\mathrm{DGSch}^{op}$  to that of stable  $\infty$ -categories and thus can be extended to a functor out of the category of prestacks.

We define the category of right crystals  $\mathrm{Crys}^r(X)$  as  $\mathrm{IndCoh}(X_{\mathrm{dR}})$ . We can also reformulate this definition à la Grothendieck by saying that a right crystal on  $X$  is an object  $\mathcal{F} \in \mathrm{IndCoh}(X)$ , together with an identification

$$(0.1) \quad x^!(\mathcal{F}) \simeq y^!(\mathcal{F})$$

for every pair of infinitesimally close points  $x, y : \mathrm{Spec}(R) \rightarrow X$  satisfying (the  $\infty$ -category version of) the cocycle condition. Note that, unlike in the case of left crystals, this does give an equivalent definition of right crystals without any coconnectivity assumptions.

0.2.3. Now that the category of right crystals is defined, we can ask whether it is equivalent to that of left crystals. The answer also turns out to be yes. Namely, for any DG scheme  $X$  almost of finite type, tensoring by the dualizing complex  $\omega_X$  defines a functor

$$\Psi_X^\vee : \mathrm{QCoh}(X) \rightarrow \mathrm{IndCoh}(X)$$

that intertwines the  $*$ -pullback on quasi-coherent sheaves and the  $!$ -pullback on ind-coherent sheaves.

Although the functor  $\Psi_S^\vee$  is not an equivalence for an individual  $S$  unless  $S$  is smooth, the totality of such maps for DG schemes mapping to the de Rham prestack of  $X$  define an equivalence between left and right crystals.

Thus, just as in the case of smooth varieties, we can think that to each DG scheme  $X$  we attach the category  $\mathrm{Crys}(X)$  equipped with two “realization” functors

$$\begin{array}{ccc} & \mathrm{Crys}(X) & \\ \mathrm{oblv}_X^l \swarrow & & \searrow \mathrm{oblv}_X^r \\ \mathrm{QCoh}(X) & \xrightarrow{\Psi_X^\vee} & \mathrm{IndCoh}(X) \end{array}$$

However, in the case of non-smooth schemes, the advantages of the t-structure on  $\mathrm{Crys}(X)$  that is associated with the “right” realization become even more pronounced.

0.2.4. *Historical remark.* To the best of our knowledge, the approach to D-modules via right crystals was first suggested by A. Beilinson in the early 90’s, at the level of abelian categories.

For some time after that it was mistakenly believed that one cannot use left crystals to define D-modules, because of the incompatibility of the t-structures. However, it was explained by J. Lurie, that if one forgoes the t-structure and defines for the corresponding stable  $\infty$ -category right away, left crystals work just as well.

0.3. **The theory of crystals/D-modules.** Let us explain the formal structure of the theory, as developed in this paper, and its sequel [GL:Funct].

0.3.1. To each prestack (locally almost of finite type)  $\mathcal{Y}$ , we assign a stable  $\infty$ -category

$$\mathcal{Y} \rightsquigarrow \mathrm{Crys}(\mathcal{Y}).$$

This category has two realization functors: a left realization functor to  $\mathrm{QCoh}(\mathcal{Y})$ , and a right realization functor to  $\mathrm{IndCoh}(\mathcal{Y})$  which are related via the following commutative diagram

$$\begin{array}{ccc} & \mathrm{Crys}(\mathcal{Y}) & \\ \mathrm{oblv}_{\mathcal{Y}}^l \swarrow & & \searrow \mathrm{oblv}_{\mathcal{Y}}^r \\ \mathrm{QCoh}(\mathcal{Y}) & \xrightarrow{\Psi_{\mathcal{Y}}^{\vee}} & \mathrm{IndCoh}(\mathcal{Y}) \end{array}$$

where  $\Psi_{\mathcal{Y}}^{\vee}$  is the functor  $\mathrm{QCoh}(\mathcal{Y}) \rightarrow \mathrm{IndCoh}(\mathcal{Y})$  given by tensoring by the dualizing complex  $\omega_{\mathcal{Y}}$ .

0.3.2. The assignment of  $\mathrm{Crys}(\mathcal{Y})$  to  $\mathcal{Y}$  is functorial in a number of ways. For a map  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ , there is a pullback functor

$$f^! : \mathrm{Crys}(\mathcal{Y}_2) \rightarrow \mathrm{Crys}(\mathcal{Y}_1)$$

which is functorial in  $f$ ; i.e., this assignment gives a functor

$$\mathrm{Crys}_{\mathrm{PreStk}}^! : (\mathrm{PreStk})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}.$$

The pullback functor on D-modules is compatible with the realization functors. Namely, we have commutative diagrams

$$\begin{array}{ccc} \mathrm{Crys}(\mathcal{Y}_1) & \xleftarrow{f^!} & \mathrm{Crys}(\mathcal{Y}_2) \\ \mathrm{oblv}_{\mathcal{Y}_1}^l \downarrow & & \downarrow \mathrm{oblv}_{\mathcal{Y}_2}^l \\ \mathrm{QCoh}(\mathcal{Y}_1) & \xleftarrow{f^*} & \mathrm{QCoh}(\mathcal{Y}_2) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Crys}(\mathcal{Y}_1) & \xleftarrow{f^!} & \mathrm{Crys}(\mathcal{Y}_2) \\ \mathrm{oblv}_{\mathcal{Y}_1}^r \downarrow & & \downarrow \mathrm{oblv}_{\mathcal{Y}_2}^r \\ \mathrm{IndCoh}(\mathcal{Y}_1) & \xleftarrow{f^!} & \mathrm{IndCoh}(\mathcal{Y}_2). \end{array}$$

Furthermore, this compatibility is itself functorial in  $f$ ; i.e. we have a naturally commutative diagram of functors

$$\begin{array}{ccc} & \mathrm{Crys}_{\mathrm{PreStk}}^! & \\ \mathrm{oblv}^l \swarrow & & \searrow \mathrm{oblv}^r \\ \mathrm{QCoh}_{\mathrm{PreStk}}^* & \xrightarrow{\Psi^{\vee}} & \mathrm{IndCoh}_{\mathrm{PreStk}}^! \end{array}.$$

0.3.3. The above portion of the theory is constructed in the present paper. I.e., this paper is concerned with the assignment

$$\mathcal{Y} \rightsquigarrow \text{Crys}(\mathcal{Y})$$

and the operation of pullback. Thus, in this paper, we develop the local theory of crystals/D-modules.

However, in addition to the functor  $f^!$ , we expect to also have a pushforward functor  $f_{dR,*}$ , and the two must satisfy various compatibility relations. The latter will be carried out in [GL:Funct]. However, let us indicate the main ingredients of the combined theory:

0.3.4. For a schematic quasi-compact map  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ , there is the de Rham pushforward functor

$$f_{dR,*} : \text{Crys}(\mathcal{Y}_1) \rightarrow \text{Crys}(\mathcal{Y}_2)$$

which is functorial in  $f$ . This assignment gives another functor

$$(\text{Crys}_{dR,*})_{\text{PreStk}_{\text{sch-qc}}} : \text{PreStk}_{\text{sch-qc}} \rightarrow \text{DGCat}_{\text{cont}},$$

where  $\text{PreStk}_{\text{sch-qc}}$  is the non-full subcategory of  $\text{PreStk}$  obtained by restricting 1-morphisms to schematic quasi-compact maps.

We have the induction functor for right crystals

$$\text{ind}_{\mathcal{Y}}^r : \text{IndCoh}(\mathcal{Y}) \rightarrow \text{Crys}(\mathcal{Y}).$$

This induction functor is compatible with de Rham pushforward. Namely, we have a commutative diagram

$$\begin{array}{ccc} \text{IndCoh}(\mathcal{Y}_1) & \xrightarrow{f_*^{\text{IndCoh}}} & \text{IndCoh}(\mathcal{Y}_2) \\ \text{ind}_{\mathcal{Y}_1}^r \downarrow & & \downarrow \text{ind}_{\mathcal{Y}_2}^r \\ \text{Crys}(\mathcal{Y}_1) & \xrightarrow{f_{dR,*}} & \text{Crys}(\mathcal{Y}_2). \end{array}$$

This compatibility is itself functorial, i.e. we have a natural transformation of functors

$$(\text{IndCoh}_*)_{\text{PreStk}_{\text{sch-qc}}} \xrightarrow{\text{ind}^r} (\text{Crys}_{dR,*})_{\text{PreStk}_{\text{sch-qc}}}.$$

0.3.5. In the case when  $f$  is proper, the functors  $(f_{dR,*}, f^!)$  form an adjoint pair, and if  $f$  is smooth, the functors  $(f^![-2n], f_{dR,*})$  form an adjoint pair for  $n$  the relative dimension of  $f$ .

In general, the two functors are not adjoint, but they satisfy a base change formula. As explained in [GL:IndCoh], a way to encode the functoriality of the base change formula is to consider a category of correspondences. Namely, let  $(\text{PreStk})_{\text{corr:all,sch-qc}}$  be the  $\infty$ -category whose objects are prestacks locally of finite type and morphisms from  $\mathcal{Y}_1$  to  $\mathcal{Y}_2$  are given by correspondences

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{g} & \mathcal{Y}_1 \\ f \downarrow & & \\ & & \mathcal{Y}_2 \end{array}$$

such that  $f$  is schematic and quasi-compact, and  $g$  arbitrary. Composition in this category is given by taking Cartesian products of correspondences. A coherent base change formula for the functors  $\text{Crys}^!$  and  $\text{Crys}_{dR,*}$  is then a functor

$$\text{Crys}_{(\text{PreStk})_{\text{corr:all,sch-qc}}} : (\text{PreStk})_{\text{corr:all,sch-qc}} \rightarrow \text{DGCat}_{\text{cont}}$$

and an identification of the restriction to  $(\mathrm{PreStk})^{op}$  with  $\mathrm{Crys}_{\mathrm{PreStk}}^!$ , and the restriction to  $\mathrm{PreStk}_{\mathrm{sch-qc}}$  with  $(\mathrm{Crys}_{\mathrm{dR},*})_{\mathrm{PreStk}_{\mathrm{sch-qc}}}$ .

#### 0.4. Twistings.

0.4.1. In addition to D-modules, it is often important to consider twisted D-modules. For instance, in representation theory, the localization theorem of Beilinson and Bernstein identifies the category of representations of a reductive Lie algebra  $\mathfrak{g}$  with fixed central character  $\chi$  with the category of twisted D-modules on the flag variety  $G/B$ , with the twisting determined by  $\chi$ .

In the case of smooth varieties, the theory of twistings and twisted D-modules was introduced by Beilinson and Bernstein [BB]. Important examples of twistings are given by complex tensor powers of line bundles. For a smooth variety  $X$ , twistings form a Picard groupoid, which can be described as follows. Let  $\mathcal{T}$  be the complex of sheaves, in degrees 1 and 2, given by

$$\mathcal{T} := \Omega^1 \rightarrow \Omega^{2,cl}$$

where  $\Omega^1$  is the sheaf of 1-forms on  $X$ ,  $\Omega^{2,cl}$  is the sheaf of closed 2-forms and the map is the de Rham differential. Then the space of objects of the Picard groupoid of twistings is given by  $H^2(X, \mathcal{T})$  and, for a given object, the space of isomorphisms is  $H^1(X, \mathcal{T})$ .

0.4.2. The second part of this paper is concerned with developing the theory of twistings and twisted crystals in the derived (and, in particular, non-smooth) context. We give several equivalent reformulations of the notion of twisting and show that they are equivalent to that defined in [BB] in the case of smooth varieties.

For a prestack (almost locally of finite type)  $\mathcal{Y}$ , we define a twisting to be a  $\mathbb{G}_m$ -gerbe on the de Rham prestack  $\mathcal{Y}_{\mathrm{dR}}$  with a trivialization of its pullback to  $\mathcal{Y}$ . A line bundle  $\mathcal{L}$  on  $\mathcal{Y}$  gives a twisting which is the trivial gerbe on  $\mathcal{Y}_{\mathrm{dR}}$ , but the trivialization on  $\mathcal{Y}$  is given by  $\mathcal{L}$ .

Given a twisting  $T$ , the category of  $T$ -twisted crystals on  $\mathcal{Y}$  is defined as the category of sheaves (ind-coherent or quasi-coherent) on  $\mathcal{Y}_{\mathrm{dR}}$  twisted by the  $\mathbb{G}_m$ -gerbe given by  $T$ .

#### 0.5. Contents.

0.5.1. In Section 1, for a prestack  $\mathcal{Y}$ , we define the de Rham prestack  $\mathcal{Y}_{\mathrm{dR}}$  and establish some of its basic properties. Most importantly, we show that if  $\mathcal{Y}$  is locally almost of finite type then so is  $\mathcal{Y}_{\mathrm{dR}}$ .

0.5.2. In Section 2, we define left crystals as quasi-coherent sheaves on the de Rham prestack and, in the locally almost of finite type case, right crystals as ind-coherent sheaves on the de Rham prestack. The latter is well-defined because, as established in Section 1, for a prestack locally almost of finite type its de Rham prestack is also locally almost of finite type. In this case, we show that the categories of left and right crystals are equivalent. Furthermore, we prove a version of Kashiwara's lemma in this setting.

0.5.3. In Section 3, we show that the category of crystals satisfies fppf descent. We also introduce the infinitesimal groupoid of a prestack  $\mathcal{Y}$  as the Čech nerve of the natural map  $\mathcal{Y} \rightarrow \mathcal{Y}_{\mathrm{dR}}$ . Specifically, the infinitesimal groupoid of  $\mathcal{Y}$  is given by

$$(\mathcal{Y} \times \mathcal{Y})_{\mathcal{Y}}^{\wedge} \rightrightarrows \mathcal{Y}$$

where  $(\mathcal{Y} \times \mathcal{Y})_{\mathcal{Y}}^{\wedge}$  is the formal completion of  $\mathcal{Y} \times \mathcal{Y}$  along the diagonal.

In much of Section 3, we specialize to the case that  $\mathcal{Y}$  is an indscheme. Sheaves on the infinitesimal groupoid of  $\mathcal{Y}$  are sheaves on  $\mathcal{Y}$  which are equivariant with respect to the equivalence relation of infinitesimal closeness. In the case of ind-coherent sheaves, this category is



equivalent to right crystals. However, quasi-coherent sheaves on the infinitesimal groupoid are, in general, not equivalent to left crystals. We show that quasi-coherent sheaves on the infinitesimal groupoid of  $\mathcal{Y}$  are equivalent to left crystals if  $\mathcal{Y}$  is eventually coconnective or classically formally smooth. Thus, in particular, this equivalence holds in the case of classical schemes.

We also define induction functors from  $\mathrm{QCoh}(\mathcal{Y})$  and  $\mathrm{IndCoh}(\mathcal{Y})$  to crystals on  $\mathcal{Y}$ . In the case of ind-coherent sheaves the induction functor is left adjoint to the forgetful functor, and we have that the category of right crystals is equivalent to the category of modules over the corresponding monad. The analogous result is true for  $\mathrm{QCoh}$  and left crystals in the case that  $\mathcal{Y}$  is eventually coconnective. Furthermore, as a functor, we identify these monads as being integral transforms for the kernel given by the sheaf of differential operators.

0.5.4. In Section 4, we show that the category of crystals has two natural t-structures: one compatible with the left realization to  $\mathrm{QCoh}$  and another compatible with the right realization to  $\mathrm{IndCoh}$ . In the case of a quasi-compact DG scheme, the two t-structures differ by a bounded amplitude.

We also show that for a quasi-compact DG scheme with affine diagonal, the category of crystals is equivalent to the derived category of its heart with respect to the right t-structure. In the case of a smooth classical scheme, we identify this abelian category with the category of modules over the algebra of differential operators.

In particular, by a standard devissage argument, we deduce that for a quasi-compact DG scheme, the heart of the category of crystals with respect to the right t-structure is Noetherian and has finite cohomological dimension.

0.5.5. In Section 5, we define the Picard groupoid of twistings on a prestack  $\mathcal{Y}$  as that of  $\mathbb{G}_m$ -gerbes on the de Rham prestack  $\mathcal{Y}_{\mathrm{dR}}$  which are trivialized on  $\mathcal{Y}$ . We then give several equivalent reformulations of this definition. For instance, using a version of the exponential map, we show that the Picard groupoid of twistings is equivalent to that of  $\mathbb{G}_a$ -gerbes on the de Rham prestack  $\mathcal{Y}_{\mathrm{dR}}$  which are trivialized on  $\mathcal{Y}$ . In particular, this makes twistings naturally a  $k$ -linear Picard groupoid.

Furthermore, using the description of twistings in terms of  $\mathbb{G}_a$ -gerbes, we identify this Picard groupoid as

$$\Omega^{\infty-2} \ker(H_{\mathrm{dR}}(\mathcal{Y}) \rightarrow H(\mathcal{Y}))$$

where  $H_{\mathrm{dR}}(Y)$  is the de Rham cohomology of  $\mathcal{Y}$ , and  $H(\mathcal{Y})$  is the coherent cohomology of  $\mathcal{Y}$ . In particular, for a smooth classical scheme, this shows that this notion of twisting agrees with that defined in [BB].

We also show that for a quasi-compact DG scheme  $X$ , twistings can be defined in terms of ind-coherent sheaves. Namely, we show that the invertible objects in  $\mathrm{IndCoh}(X)$  with respect to the  $!$ -tensor product are equivalent to line bundles on  $X$ . Using this fact, we identify twistings on an indscheme  $\mathcal{X}$  locally of finite type with central extensions of its infinitesimal groupoid.

0.5.6. In Section 6, we define the category of twisted crystals and establish its basic properties. In particular, we show that most results about crystals carry over to the twisted setting.

**0.6. Conventions and notation.** Our conventions follow closely those of [GL:IndSch]. Let us recall the most essential ones.

0.6.1. *The ground field.* Throughout the paper we will be working over a fixed ground field  $k$ . To be consistent with [GL:IndCoh], we assume that  $\text{char}(k) = 0$ . Most of the results of this paper do not depend on this assumption. The main place where this assumption is used is in the discussion of twistings using the exponential map in Sect. 5.4 and Sect. 5.5. However, even though most of the results of this paper carry over to characteristic  $p$ , it is probably better to consider a different notion in that setting that allows for characteristic 0 deformations as well, as one usually does in the study of crystalline cohomology in finite characteristic. In the language of [Gr], the theory we develop is that of sheaves on the infinitesimal site rather than the crystalline one.

0.6.2.  *$\infty$ -categories.* By an  $\infty$ -category we shall always mean an  $(\infty, 1)$ -category. By a slight abuse of language, we will sometimes refer to “categories” when we actually mean  $\infty$ -categories. Our usage of  $\infty$ -categories is model independent, but we have in mind their realization as quasi-categories. The basic reference for  $\infty$ -categories as quasi-categories is [Lu0].

By  $\infty\text{-Grpd}$  we denote the  $\infty$ -category of  $\infty$ -groupoids, which is the same as the category  $\mathcal{S}$  of spaces in the notation of [Lu0].

For an  $\infty$ -category  $\mathbf{C}$ , and  $x, y \in \mathbf{C}$ , we shall denote by  $\text{Maps}_{\mathbf{C}}(x, y) \in \infty\text{-Grpd}$  the corresponding mapping space. By  $\text{Hom}_{\mathbf{C}}(x, y)$  we denote the set  $\pi_0(\text{Maps}_{\mathbf{C}}(x, y))$ , i.e., what is denoted  $\text{Hom}_{h\mathbf{C}}(x, y)$  in [Lu0].

A stable  $\infty$ -category  $\mathbf{C}$  is naturally enriched in spectra. In this case, for  $x, y \in \mathbf{C}$ , we shall denote by  $\text{Maps}_{\mathbf{C}}(x, y)$  the spectrum of maps from  $x$  to  $y$ . In particular, we have that  $\text{Maps}_{\mathbf{C}}(x, y) = \Omega^{\infty} \text{Maps}_{\mathbf{C}}(x, y)$ .

When working in a fixed  $\infty$ -category  $\mathbf{C}$ , for two objects  $x, y \in \mathbf{C}$ , we shall call a point of  $\text{Maps}_{\mathbf{C}}(x, y)$  an *isomorphism* what is in [Lu0] is called an *equivalence*. I.e., an isomorphism is a map that admits a homotopy inverse. We reserve the word “equivalence” to mean a (homotopy) equivalence between  $\infty$ -categories.

0.6.3. *DG categories.* Our conventions regarding DG categories follow [GL:IndCoh], Sect. 0.6.4. By a DG category we shall understand a presentable DG category over  $k$ . Unless specified otherwise, we will only consider continuous functors between DG categories (i.e., exact functors that commute with direct sums, or equivalently, with all colimits). In other words, we will be working in the category  $\text{DGCat}_{\text{cont}}$  in the notation of [GL:DG].<sup>2</sup>

We let  $\text{Vect}$  denote the DG category of complexes of  $k$ -vector space. The category  $\text{DGCat}_{\text{cont}}$  has a natural symmetric monoidal structure, for which  $\text{Vect}$  is the unit.

For a DG category  $\mathbf{C}$  equipped with a t-structure, we denote by  $\mathbf{C}^{\leq n}$  (resp.,  $\mathbf{C}^{\geq m}$ ,  $\mathbf{C}^{\leq n, \geq m}$ ) the corresponding full subcategory of  $\mathbf{C}$  spanned by objects  $x$ , such that  $H^i(x) = 0$  for  $i > n$  (resp.,  $i < m$ ,  $(i > n) \wedge (i < m)$ ). The inclusion  $\mathbf{C}^{\leq n} \hookrightarrow \mathbf{C}$  admits a right adjoint denoted by  $\tau^{\leq n}$ , and similarly, for the other categories.

There is a fully faithful functor from  $\text{DGCat}_{\text{cont}}$  to that of stable  $\infty$ -categories and continuous exact functors. A stable  $\infty$ -category obtained in this way is enriched over the category  $\text{Vect}$ . Thus, we shall often think of the spectrum  $\text{Maps}_{\mathbf{C}}(x, y)$  as an object of  $\text{Vect}$ ; the former is obtained from the latter by the Dold-Kan correspondence.

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<sup>2</sup>One can replace  $\text{DGCat}_{\text{cont}}$  by (the equivalent)  $(\infty, 1)$ -category of stable presentable  $\infty$ -categories tensored over  $\text{Vect}$ , with colimit-preserving functors.

0.6.4. *(Pre)stacks and DG schemes.* Our conventions regarding (pre)stacks and DG schemes follow [GL:Stacks]:

Let  $\mathrm{DGSch}^{\mathrm{aff}}$  denote the  $\infty$ -category opposite to that of *connective* commutative DG algebras over  $k$ .

The category  $\mathrm{PreStk}$  of prestacks is by definition that of all functors

$$(\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-Grpd}.$$

Let  $<\infty\mathrm{DGSch}^{\mathrm{aff}}$  be the full subcategory of  $\mathrm{DGSch}^{\mathrm{aff}}$  given by eventually coconnective objects.

Recall that an eventually coconnective affine DG scheme  $S = \mathrm{Spec}(A)$  is *almost of finite type* if

- $H^0(A)$  is finite type over  $k$ .
- Each  $H^i(A)$  is finitely generated as a module over  $H^0(A)$ .

Let  $<\infty\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}$  denote the full subcategory of  $<\infty\mathrm{DGSch}^{\mathrm{aff}}$  consisting of schemes almost of finite type, and let  $\mathrm{PreStk}_{\mathrm{laft}}$  be the category of all functors

$$<\infty(\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-Grpd}.$$

As explained in [GL:Stacks, Sect. 1.3.11],  $\mathrm{PreStk}_{\mathrm{laft}}$  is naturally a subcategory of  $\mathrm{PreStk}$  via a suitable Kan extension.

In order to apply the formalism of ind-coherent sheaves developed in [GL:IndCoh], we assume that the prestacks we consider are locally almost of finite type for most of this paper. We will explicitly indicate when this is not the case.

0.6.5. *Reduced rings.* Let  $(^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}})^{op} \subset (\mathrm{DGSch}^{\mathrm{aff}})^{op}$  denote the category of reduced discrete rings. The inclusion functor has a natural left adjoint

$$cl, \mathrm{red}(-) : (\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow (^{\mathrm{red}}\mathrm{Sch}^{\mathrm{aff}})^{op}$$

given by

$$S \mapsto H^0(S) / \mathrm{nilp}(H^0(S))$$

where  $\mathrm{nilp}(H^0(S))$  is the ideal of nilpotent elements in  $H^0(S)$ .

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## 1. THE DE RHAM PRESTACK

For a prestack  $\mathcal{Y}$ , crystals are defined as sheaves (quasi-coherent or ind-coherent) on the de Rham prestack  $\mathcal{Y}_{\mathrm{dR}}$  of  $\mathcal{Y}$ . In this section, we define the functor  $\mathcal{Y} \mapsto \mathcal{Y}_{\mathrm{dR}}$  and establish a number of its basic properties.

Most importantly, we will show that if  $\mathcal{Y}$  is locally almost of finite type, then so is  $\mathcal{Y}_{\mathrm{dR}}$ . In this case, we will also show that  $\mathcal{Y}_{\mathrm{dR}}$  is classical, i.e., it can be studied entirely within the realm of “classical” algebraic geometry without reference to derived rings.

As the reader might find this section particularly abstract, it might be a good strategy to skip it on first pass, and return to it when necessary when assertions established here are applied to crystals.

### 1.1. Definition and basic properties.

1.1.1. Let  $\mathcal{Y}$  be an object of  $\text{PreStk}$ . We define the de Rham prestack of  $\mathcal{Y}$ ,  $\mathcal{Y}_{\text{dR}} \in \text{PreStk}$  as

$$(1.1) \quad \mathcal{Y}_{\text{dR}}(S) := \mathcal{Y}^{(cl, red)}(S)$$

for  $S \in \text{DGSch}^{\text{aff}}$ .

1.1.2. More abstractly, we can rewrite

$$\mathcal{Y}_{\text{dR}} := \text{RKE}_{\text{redSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Y}^{(cl, red)}),$$

where  $\mathcal{Y}^{(cl, red)} := \mathcal{Y}|_{\text{redSch}^{\text{aff}}}$  is the restriction of  $\mathcal{Y}$  to reduced classical affine schemes, and

$$\text{RKE}_{\text{redSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}$$

is the right Kan extension of a functor along the inclusion  $\text{redSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}$ .

1.1.3. The following (obvious) observation will be useful in the sequel.

**Lemma 1.1.4.** *The functor  $\text{dR} : \text{PreStk} \rightarrow \text{PreStk}$  commutes with limits and colimits.*

As a consequence, we obtain:

**Corollary 1.1.5.** *The functor  $\text{dR} : \text{PreStk} \rightarrow \text{PreStk}$  is the left Kan extension of the functor*

$$\text{dR}|_{\text{DGSch}^{\text{aff}}} : \text{DGSch}^{\text{aff}} \rightarrow \text{PreStk}$$

*along  $\text{DGSch}^{\text{aff}} \hookrightarrow \text{PreStk}$ .*

1.1.6. Furthermore, we have:

**Lemma 1.1.7.** *The functor  $\text{dR}|_{\text{DGSch}^{\text{aff}}} : \text{DGSch}^{\text{aff}} \rightarrow \text{PreStk}$  is isomorphic to the left Kan of the functor*

$$\text{dR}|_{\text{redSch}^{\text{aff}}} : \text{redSch}^{\text{aff}} \rightarrow \text{PreStk}$$

*along  $\text{redSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}$ .*

*Proof.* For any target category  $\mathbf{D}$  and any functor  $\Phi : \text{DGSch}^{\text{aff}} \rightarrow \mathbf{D}$ , the map

$$\text{LKE}_{\text{redSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\Phi|_{\text{redSch}^{\text{aff}}}) \rightarrow \Phi$$

is an isomorphism if and only if  $\Phi$  factors through the functor

$$\text{DGSch}^{\text{aff}} \rightarrow \text{redSch}^{\text{aff}} : S \mapsto \mathcal{Y}^{(cl, red)}(S),$$

which is the case for  $\mathbf{D} = \text{PreStk}$  and  $\Phi$  the functor  $S \mapsto S_{\text{dR}}$ . □

1.1.8. Let  $\mathbf{C}_1 \subset \mathbf{C}_2$  be a pair of categories from the following full subcategories of  $\text{PreStk}$ :

$$\text{redSch}^{\text{aff}}, \text{Sch}^{\text{aff}}, \text{DGSch}^{\text{aff}}, \text{DGSch}_{\text{qc-qs}}, \text{DGSch}, \text{PreStk}$$

From Lemma 1.1.7 and Corollary 1.1.5 we obtain:

**Corollary 1.1.9.** *The functor  $\mathbf{C}_2 \rightarrow \text{PreStk}$  given by  $\text{dR}|_{\mathbf{C}_2}$  is isomorphic to the left Kan extension along  $\mathbf{C}_1 \hookrightarrow \mathbf{C}_2$  of the functor  $\text{dR}|_{\mathbf{C}_1} : \mathbf{C}_1 \rightarrow \text{PreStk}$ .*

### 1.2. Relation between $\mathcal{Y}$ and $\mathcal{Y}_{\text{dR}}$ .

1.2.1. The functor  $\text{dR} : \text{PreStk} \rightarrow \text{PreStk}$  comes equipped with a natural transformation

$$p_{\text{dR}} : \text{Id} \rightarrow \text{dR},$$

i.e., for every  $\mathcal{Y} \in \text{PreStk}$  we have a canonical map

$$p_{\text{dR}, \mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y}_{\text{dR}}.$$

1.2.2. Let  $\mathcal{Y}^\bullet/\mathcal{Y}_{\text{dR}}$  be the Čech nerve of  $p_{\text{dR},\mathcal{Y}}$ , regarded as a simplicial object of  $\text{PreStk}$ . It is augmented by  $\mathcal{Y}_{\text{dR}}$ .

Note that each  $\mathcal{Y}^i/\mathcal{Y}_{\text{dR}}$  is the formal completion of  $\mathcal{Y}^i$  along the main diagonal. (We refer the reader to [GL:IndSch], Sect. 6.1.1 for our conventions regarding formal completions).

We have a canonical map

$$(1.2) \quad |\mathcal{Y}^\bullet/\mathcal{Y}_{\text{dR}}| \rightarrow \mathcal{Y}_{\text{dR}}.$$

1.2.3. *Classically formally smooth prestacks.* We shall say that a prestack  $\mathcal{Y}$  is classically formally smooth, if for  $S \in \text{DGSch}^{\text{aff}}$ , the map

$$\text{Maps}(S, \mathcal{Y}) \rightarrow \text{Maps}^{(cl, red)}(S, \mathcal{Y})$$

induces a surjection on  $\pi_0$ .

The following results from the definitions:

**Lemma 1.2.4.** *If  $\mathcal{Y}$  is classically formally smooth, the map*

$$|\mathcal{Y}^\bullet/\mathcal{Y}_{\text{dR}}| \rightarrow \mathcal{Y}_{\text{dR}}$$

*is an isomorphism in  $\text{PreStk}$ .*

1.3. **The locally almost of finite type case.** Recall that  $\text{PreStk}$  contains a full subcategory  $\text{PreStk}_{\text{laft}}$  of prestacks locally almost of finite type, see [GL:Stacks], Sect. 1.3.9.

1.3.1. The following observation will play an important role in this paper.

**Proposition 1.3.2.** *Assume that  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$ . Then:*

- (a)  $\mathcal{Y}_{\text{dR}} \in \text{PreStk}_{\text{laft}}$ .
- (b)  $\mathcal{Y}_{\text{dR}}$  is classical, i.e., belongs to the full subcategory  $^{cl}\text{PreStk} \subset \text{PreStk}$ .

1.3.3. *Proof of point (a).*

We need to verify two properties:

- (i)  $\mathcal{Y}_{\text{dR}}$  is convergent (see [GL:Stacks], Sect. 1.21] for the definition);
- (ii) Each truncation  $\leq^n(\mathcal{Y}_{\text{dR}})$  is locally of finite type.

Property (i) follows tautologically; it is true for any  $\mathcal{Y} \in \text{PreStk}$ . To establish property (ii), we need to show that the functor  $\mathcal{Y}_{\text{dR}}$  takes filtered limits in  $\leq^n \text{DGSch}^{\text{aff}}$  to colimits in  $\infty\text{-Grpd}$ . Since  $\mathcal{Y}$  itself has this property, it suffices to show that the functor

$$S \mapsto {}^{cl, red}S : \text{DGSch}^{\text{aff}} \rightarrow \text{DGSch}^{\text{aff}}$$

preserves filtered limits, which is evident. □

1.3.4. *Proof of point (b).*

By Corollary 1.1.9, we need to prove that the colimit

$$\operatorname{colim}_{S \in ((\operatorname{Sch}^{\text{aff}})_{/\mathcal{Y}})^{\text{op}}} S_{\text{dR}} \in \operatorname{PreStk}$$

is classical. By part (a), the functor

$$(\operatorname{Sch}_{\text{ft}}^{\text{aff}})_{/\mathcal{Y}} \rightarrow (\operatorname{Sch}^{\text{aff}})_{/\mathcal{Y}}$$

is cofinal; hence,

$$\operatorname{colim}_{S \in ((\operatorname{Sch}_{\text{ft}}^{\text{aff}})_{/\mathcal{Y}})^{\text{op}}} S_{\text{dR}} \rightarrow \operatorname{colim}_{S \in ((\operatorname{Sch}^{\text{aff}})_{/\mathcal{Y}})^{\text{op}}} S_{\text{dR}}$$

is an isomorphism.

Therefore, since the full subcategory  ${}^{\text{cl}}\operatorname{PreStk} \subset \operatorname{PreStk}$  is closed under colimits, we can assume without loss of generality that  $\mathcal{Y}$  is a classical affine scheme of finite type.

More generally, we will show that for  $X \in \operatorname{DGSch}_{\text{aft}}^{\text{aff}}$ , the prestack  $X_{\text{dR}}$  is classical. Let  $i : X \hookrightarrow Z$  be a closed embedding, where  $Z$  is a *smooth* classical affine scheme of finite type. Let  $Y$  denote the formal completion of  $Z$  along  $X$ . The map  $X \rightarrow Y$  induces an isomorphism  $X_{\text{dR}} \rightarrow Y_{\text{dR}}$ . Hence, it suffices to show that  $Y_{\text{dR}}$  is classical.

Consider  $Y^\bullet/Y_{\text{dR}}$  (see Sect. 1.2.3 above). Note that  $Y$  is formally smooth, since  $Z$  is (see [GL:IndSch, Sect. 8.1]). In particular,  $Y$  is classically formally smooth. Since the subcategory  ${}^{\text{cl}}\operatorname{PreStk} \subset \operatorname{PreStk}$  is closed under colimits and by Lemma 1.2.4, it suffices to show that each term  $Y^i/Y_{\text{dR}}$  is classical as a prestack.

Note that  $Y^i/Y_{\text{dR}}$  is isomorphic to the formal completion of  $Z^i$  along the diagonally embedded copy of  $X$ . Hence,  $Y^i/Y_{\text{dR}}$  is classical by [GL:IndSch, Proposition 6.7.2].  $\square$

1.3.5. From Proposition 1.3.2 we obtain:

**Corollary 1.3.6.** *Let  $\mathbf{C}_1 \subset \mathbf{C}_2$  be any of the following full subcategories of  $\operatorname{DGSch}^{\text{aff}}$ :*

$$\operatorname{Sch}_{\text{ft}}^{\text{aff}}, {}^{<\infty}\operatorname{DGSch}_{\text{ft}}^{\text{aff}}, \operatorname{DGSch}_{\text{aft}}^{\text{aff}}, \operatorname{Sch}^{\text{aff}}, \operatorname{DGSch}^{\text{aff}}.$$

*Then for  $\mathcal{Y} \in \operatorname{PreStk}_{\text{laft}}$ , the functor*

$$(\mathbf{C}_1)_{/\mathcal{Y}_{\text{dR}}} \rightarrow (\mathbf{C}_2)_{/\mathcal{Y}_{\text{dR}}}$$

*is cofinal.*

*Proof.* It suffices to prove the assertion for the inclusions

$$\operatorname{Sch}_{\text{ft}}^{\text{aff}} \hookrightarrow \operatorname{Sch}^{\text{aff}} \hookrightarrow \operatorname{DGSch}^{\text{aff}}.$$

For right arrow, the assertion follows from point (b) of the proposition, and for the left arrow from point (a) of the proposition.  $\square$

1.3.7. Now, consider the following full subcategories

$$(1.3) \quad {}^{red}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}, \mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}, \mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}, \mathrm{DGSch}_{\mathrm{aft}}, \mathrm{DGSch}_{\mathrm{laft}}, \mathrm{PreStk}_{\mathrm{laft}}.$$

of the categories appearing in Sect. 1.1.8.

**Corollary 1.3.8.** *The restriction of the functor  $\mathrm{dR}$  to  $\mathrm{PreStk}_{\mathrm{laft}}$  is isomorphic to the left Kan extension of this functor to  $\mathbf{C}$ , where  $\mathbf{C}$  is one of the subcategories in (1.3).*

*Proof.* It suffices to prove the corollary for  $\mathbf{C} = {}^{red}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$ . By Corollary 1.1.9, it is enough to show that for  $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ , the functor

$$({}^{red}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{Y}} \rightarrow ({}^{red}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$$

is cofinal.

By Proposition 1.3.2(a), the functor

$$(\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{Y}} \rightarrow (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$$

is cofinal. Now, the assertion follows from the fact that the inclusion  ${}^{red}\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$  admits a right adjoint.  $\square$

## 2. DEFINITION OF CRYSTALS

In this section we will define left crystals (for arbitrary objects of  $\mathrm{PreStk}$ ), and right crystals for objects of  $\mathrm{PreStk}_{\mathrm{laft}}$ . We will show that in the latter case, the two theories are equivalent.

### 2.1. Left crystals.

2.1.1. For  $\mathcal{Y} \in \mathrm{PreStk}$  we define

$$\mathrm{Crys}^l(\mathcal{Y}) := \mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}}).$$

I.e.,

$$\mathrm{Crys}^l(\mathcal{Y}) = \lim_{S \in (\mathrm{DGSch}_{/\mathcal{Y}_{\mathrm{dR}}}^{\mathrm{aff}})^{op}} \mathrm{QCoh}(S).$$

Informally, an object  $\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})$  is an assignment of a quasi-coherent sheaf  $\mathcal{F}_S \in \mathrm{QCoh}(S)$  for every affine DG scheme  $S \in \mathrm{DGSch}^{\mathrm{aff}}$  with a map  ${}^{red,cl}S \rightarrow \mathcal{Y}$ , as well as an isomorphism

$$f^*(\mathcal{F}_S) \simeq \mathcal{F}_{S'} \in \mathrm{QCoh}(S')$$

for every morphism  $f : S' \rightarrow S$  of affine DG schemes.

2.1.2. More functorially, let  $\mathrm{Crys}_{\mathrm{PreStk}}^l$  denote the functor  $(\mathrm{PreStk})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$  defined as

$$\mathrm{Crys}_{\mathrm{PreStk}}^l := \mathrm{QCoh}_{\mathrm{PreStk}} \circ \mathrm{dR},$$

where

$$\mathrm{QCoh}_{\mathrm{PreStk}} : (\mathrm{PreStk})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$$

is the functor which assigns to a prestack the corresponding category of quasi-coherent sheaves [GL:QCoh, Sect. 1.1.5].

For a map  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  in  $\mathrm{PreStk}$ , let  $f^\dagger$  denote the corresponding pullback functor

$$\mathrm{Crys}^l(\mathcal{Y}_2) \rightarrow \mathrm{Crys}^l(\mathcal{Y}_1).$$

By construction, if  $f$  induces an isomorphism of the underlying reduced classical prestacks  ${}^{cl,red}\mathcal{Y}_1 \rightarrow {}^{cl,red}\mathcal{Y}_2$ , then it induces an isomorphism of de Rham prestacks  $\mathcal{Y}_{1,\mathrm{dR}} \rightarrow \mathcal{Y}_{2,\mathrm{dR}}$  and in particular  $f^\dagger$  is an equivalence.

2.1.3. Recall that the functor  $\mathrm{QCoh}_{\mathrm{PreStk}} : (\mathrm{PreStk})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$  is by definition the right Kan extension of the functor

$$\mathrm{QCoh}_{\mathrm{DGSch}^{\mathrm{aff}}} : (\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$$

along  $\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}$ .

In particular, it takes colimits in  $\mathrm{PreStk}$  to limits in  $\mathrm{DGCat}_{\mathrm{cont}}$ . Therefore, by Corollary 1.1.9, for  $\mathcal{Y} \in \mathrm{PreStk}$  we obtain:

**Corollary 2.1.4.** *Let  $\mathbf{C}$  be any of the categories from the list of Sect. 1.1.8. Then for  $\mathcal{Y} \in \mathrm{PreStk}$ , the functor*

$$\mathrm{Crys}^l(\mathcal{Y}) \rightarrow \lim_{X \in (\mathbf{C}/\mathcal{Y})^{op}} \mathrm{Crys}^l(X)$$

*is an equivalence.*

Informally, this corollary says that the data of an object  $\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})$  is equivalent to that of  $\mathcal{M}_S \in \mathrm{Crys}^l(S)$  for every  $S \in \mathbf{C}/\mathcal{Y}$ , and for every  $f : S' \rightarrow S$ , an isomorphism

$$f^\dagger(\mathcal{M}_S) \simeq \mathcal{M}_{S'} \in \mathrm{Crys}^l(S').$$

2.1.5. Recall the natural transformation  $p_{\mathrm{dR}} : \mathrm{Id} \rightarrow \mathrm{dR}$ . It induces a natural transformation

$$\mathbf{oblv}^l : \mathrm{Crys}_{\mathrm{PreStk}}^l \rightarrow \mathrm{QCoh}_{\mathrm{PreStk}}.$$

I.e., for every  $\mathcal{Y} \in \mathrm{PreStk}$ , we have a functor

$$(2.1) \quad \mathbf{oblv}_{\mathcal{Y}}^l : \mathrm{Crys}^l(\mathcal{Y}) \rightarrow \mathrm{QCoh}(\mathcal{Y}),$$

and for every morphism  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ , a commutative diagram:

$$(2.2) \quad \begin{array}{ccc} \mathrm{Crys}^l(\mathcal{Y}_1) & \xrightarrow{\mathbf{oblv}_{\mathcal{Y}_1}^l} & \mathrm{QCoh}(\mathcal{Y}_1) \\ f^\dagger \uparrow & & \uparrow f^* \\ \mathrm{Crys}^l(\mathcal{Y}_2) & \xrightarrow{\mathbf{oblv}_{\mathcal{Y}_2}^l} & \mathrm{QCoh}(\mathcal{Y}_2). \end{array}$$

2.1.6. Recall the simplicial object  $\mathcal{Y}^\bullet/\mathcal{Y}_{\mathrm{dR}}$  of Sect. 1.2.2.

From Lemma 1.2.4 we obtain:

**Corollary 2.1.7.** *If  $\mathcal{Y}$  is classically formally smooth, then the functor*

$$\mathrm{Crys}^l(\mathcal{Y}) \rightarrow \mathrm{Tot}(\mathrm{QCoh}(\mathcal{Y}^\bullet/\mathcal{Y}_{\mathrm{dR}}))$$

*is an equivalence.*

**2.2. Left crystals on prestacks locally almost of finite type.** For the rest of this section, unless specified otherwise, we will restrict ourselves to the subcategory  $\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$ .

So, unless explicitly stated otherwise, by a prestack/DG scheme/affine DG scheme, we shall mean one which is locally almost of finite type.

Let  $\mathrm{Crys}_{\mathrm{PreStk}_{\mathrm{laft}}}^l$  denote the restriction of  $\mathrm{Crys}_{\mathrm{PreStk}}^l$  to  $\mathrm{PreStk}_{\mathrm{laft}} \subset \mathrm{PreStk}$ .



2.2.1. The next corollary says that we “do not need to know” about schemes of infinite type as well as derived algebraic geometry in order to define  $\text{Crys}^l(\mathcal{Y})$  for  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$ . In other words, to define crystals on a prestack locally almost of finite type, we can stay within the world of classical affine schemes of finite type.

Indeed, from Corollary 1.3.6 we obtain:

**Corollary 2.2.2.** *Let  $\mathbf{C}$  be one of the full subcategories*

$$\text{Sch}_{\text{ft}}^{\text{aff}}, {}^{<\infty}\text{DGSch}_{\text{ft}}^{\text{aff}}, \text{DGSch}_{\text{aft}}^{\text{aff}}, \text{Sch}^{\text{aff}}$$

*of  $\text{DGSch}^{\text{aff}}$ . Then for  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$  the natural functor*

$$\text{Crys}^l(\mathcal{Y}) \rightarrow \lim_{S \in (\mathbf{C}/\mathcal{Y}_{\text{dR}})^{\text{op}}} \text{QCoh}(S)$$

*is an equivalence.*

2.2.3. Recall that according to Corollary 2.1.4, the category  $\text{Crys}^l(\mathcal{Y})$  can be recovered from the functor

$$\text{Crys}^l : \mathbf{C}/\mathcal{Y} \rightarrow \text{DGCat}_{\text{cont}}$$

where  $\mathbf{C}$  is any one of the categories

$${}^{\text{red}}\text{Sch}^{\text{aff}}, \text{Sch}^{\text{aff}}, \text{DGSch}^{\text{aff}}, \text{DGSch}_{\text{qc-qs}}, \text{DGSch} \subset \text{PreStk}.$$

We now claim that the above categories can be also replaced by their full subcategories in the list (1.3):

$${}^{\text{red}}\text{Sch}_{\text{ft}}^{\text{aff}}, \text{Sch}_{\text{ft}}^{\text{aff}}, \text{DGSch}_{\text{aft}}^{\text{aff}}, \text{DGSch}_{\text{aft}}, \text{DGSch}_{\text{laft}}, \text{PreStk}_{\text{laft}}.$$

**Corollary 2.2.4.** *For  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$  and  $\mathbf{C}$  being one of the categories in (1.3), the functor*

$$\text{Crys}^l(\mathcal{Y}) \rightarrow \lim_{X \in (\mathbf{C}/\mathcal{Y})^{\text{op}}} \text{Crys}^l(X)$$

*is an equivalence.*

Indeed, this follows from Corollary 1.3.8.

Informally, the above corollary says that an object  $\mathcal{M} \in \text{Crys}^l(\mathcal{Y})$  can be recovered from an assignment of  $\mathcal{M}_S \in \text{Crys}^l(S)$  for every  $S \in \mathbf{C}/\mathcal{Y}$ , and for every  $f : S' \rightarrow S$  of an isomorphism

$$f^\dagger(\mathcal{M}_S) \simeq \mathcal{M}_{S'} \in \text{Crys}^l(S').$$

2.2.5. Consider again the functor

$$\mathbf{oblv}_{\mathcal{Y}}^l : \text{Crys}^l(\mathcal{Y}) \rightarrow \text{QCoh}(\mathcal{Y})$$

of (2.1). We have:

**Lemma 2.2.6.** *For  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$ , the functor  $\mathbf{oblv}_{\mathcal{Y}}^l$  is conservative.*

*Proof.* By Corollary 2.2.4 and the commutativity of (2.2), we can assume without loss of generality that  $\mathcal{Y} = X$  is an affine DG scheme locally almost of finite type. Let  $i : X \rightarrow Z$  be a closed embedding of  $X$  into a smooth classical finite type scheme  $Z$ , and let  $Y$  be the formal completion of  $Z$  along  $X$ . In this case,  $Y_{\text{dR}} \simeq X_{\text{dR}}$ . Since  $Y$  is formally smooth (and, in particular, classically formally smooth), by Corollary 2.1.7, the functor

$$\mathbf{oblv}^l : \text{Crys}^l(Y) \rightarrow \text{QCoh}(Y)$$

is conservative.

The lemma now follows from the fact that the restriction functor  $\mathrm{QCoh}(Y) \rightarrow \mathrm{QCoh}(X)$  is conservative (the latter is established, e.g., in the last paragraph of the proof of [GL:IndSch, Proposition 7.1.3]).  $\square$

### 2.3. Right crystals.

2.3.1. Recall that  $\mathrm{PreStk}_{\mathrm{laft}}$  can be alternatively viewed as the category of all functors

$$(<^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-Grpd},$$

see [GL:Stacks], Sect. 1.3.11.

Furthermore, recall the functor

$$\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}^! : (\mathrm{PreStk}_{\mathrm{laft}})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$$

of [GL:IndCoh, Sect. 9.1.5], which is defined as the right Kan extension of the corresponding functor

$$\mathrm{IndCoh}_{<^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}}^! : (<^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}.$$

In particular, the functor  $\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}^!$  takes colimits in  $\mathrm{PreStk}_{\mathrm{laft}}$  to limits in  $\mathrm{DGCat}_{\mathrm{cont}}$ .

2.3.2. We define the functor

$$\mathrm{Crys}_{\mathrm{PreStk}_{\mathrm{laft}}}^r : (\mathrm{PreStk}_{\mathrm{laft}})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$$

as the composite

$$\mathrm{Crys}_{\mathrm{PreStk}_{\mathrm{laft}}}^r := \mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}^! \circ \mathrm{dR}.$$

In the above formula, Proposition 1.3.2(a) is used to make sure that  $\mathrm{dR}$  is defined as a functor  $\mathrm{PreStk}_{\mathrm{laft}} \rightarrow \mathrm{PreStk}_{\mathrm{laft}}$ .

*Remark 2.3.3.* In defining  $\mathrm{Crys}_{\mathrm{PreStk}_{\mathrm{laft}}}^r$  we “do not need to know” about schemes of infinite type: we can define the endo-functor  $\mathrm{dR} : \mathrm{PreStk}_{\mathrm{laft}} \rightarrow \mathrm{PreStk}_{\mathrm{laft}}$  directly by the formula

$$\mathrm{Maps}(S, \mathcal{Y}_{\mathrm{dR}}) = \mathrm{Maps}({}^{red, cl} S, \mathcal{Y})$$

for  $S \in <^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}$ .

2.3.4. For a map  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  in  $\mathrm{PreStk}_{\mathrm{laft}}$ , we shall denote by  $f^!$  the corresponding functor  $\mathrm{Crys}^r(\mathcal{Y}_2) \rightarrow \mathrm{Crys}^r(\mathcal{Y}_1)$ .

If  $f$  induces an equivalence  ${}^{cl, red} \mathcal{Y}_1 \rightarrow {}^{cl, red} \mathcal{Y}_2$ , then the map  $\mathcal{Y}_{1, \mathrm{dR}} \rightarrow \mathcal{Y}_{2, \mathrm{dR}}$  is an equivalence, and in particular, so is  $f^!$ .

2.3.5. By definition, for  $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ , we have:

$$\mathrm{Crys}^r(\mathcal{Y}) = \lim_{S \in ((<^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{Y}_{\mathrm{dR}}})^{op}} \mathrm{IndCoh}(S).$$

Informally, an object  $\mathcal{M} \in \mathrm{Crys}^r(\mathcal{Y})$  is an assignment for every  $S \in <^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}$  and a map  ${}^{red, cl} S \rightarrow \mathcal{Y}$  of an object  $\mathcal{F}_S \in \mathrm{IndCoh}(S)$ , and for every  $f : S' \rightarrow S$  of an isomorphism

$$f^!(\mathcal{F}_S) \simeq \mathcal{F}_{S'} \in \mathrm{IndCoh}(S').$$

2.3.6. As in Sect. 2.2.1, we “do not need to know” about DG schemes in order to recover  $\text{Crys}^r(\mathcal{Y})$ :

**Corollary 2.3.7.** *For  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$ , the functor*

$$\text{Crys}^r(\mathcal{Y}) \rightarrow \lim_{S \in ((\text{Sch}_{\text{ft}}^{\text{aff}})_{/\mathcal{Y}_{\text{dR}}})^{\text{op}}} \text{IndCoh}(S)$$

*is an equivalence.*

This follows readily from Corollary 1.3.6.

Informally, the above corollary says that an  $\mathcal{M} \in \text{Crys}^r(\mathcal{Y})$  can be recovered from an assignment for every  $S \in \text{Sch}_{\text{ft}}^{\text{aff}}$  and a map  ${}^{\text{red}, \text{cl}} S \rightarrow \mathcal{Y}$  of an object  $\mathcal{F}_S \in \text{IndCoh}(S)$ , and for every  $f : S' \rightarrow S$  of an isomorphism

$$f^!(\mathcal{F}_S) \simeq \mathcal{F}_{S'} \in \text{IndCoh}(S').$$

2.3.8. Furthermore, the analogue of Corollary 2.2.4 holds for right crystals as well:

**Corollary 2.3.9.** *Let  $\mathbf{C}$  be any of the categories from (1.3). Then the functor*

$$\text{Crys}^r(\mathcal{Y}) \rightarrow \lim_{X \in (\mathbf{C}_{/\mathcal{Y}})^{\text{op}}} \text{Crys}^r(X)$$

*is an equivalence.*

This follows from Corollary 1.3.8.

Informally, this corollary says that we can recover an object  $\mathcal{M} \in \text{Crys}^r(\mathcal{Y})$  from an assignment of  $\mathcal{M}_S \in \text{Crys}^r(S)$  for every  $S \in \mathbf{C}_{/\mathcal{Y}}$ , and for every  $f : S' \rightarrow S$  of an isomorphism

$$f^!(\mathcal{M}_S) \simeq \mathcal{M}_{S'} \in \text{Crys}^r(S').$$

2.3.10. The natural transformation  $p_{\text{dR}} : \text{Id} \rightarrow \text{dR}$  induces a natural transformation

$$\mathbf{oblv}^r : \text{Crys}_{\text{PreStk}_{\text{laft}}}^r \rightarrow \text{IndCoh}_{\text{PreStk}_{\text{laft}}}.$$

I.e., for every  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$ , we have a functor

$$\mathbf{oblv}_{\mathcal{Y}}^r : \text{Crys}^r(\mathcal{Y}) \rightarrow \text{IndCoh}(\mathcal{Y}),$$

and for every morphism  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ , a commutative diagram:

$$(2.3) \quad \begin{array}{ccc} \text{Crys}^r(\mathcal{Y}_1) & \xrightarrow{\mathbf{oblv}_{\mathcal{Y}_1}^r} & \text{IndCoh}(\mathcal{Y}_1) \\ f^! \uparrow & & \uparrow f^! \\ \text{Crys}^r(\mathcal{Y}_2) & \xrightarrow{\mathbf{oblv}_{\mathcal{Y}_2}^r} & \text{IndCoh}(\mathcal{Y}_2). \end{array}$$

**Lemma 2.3.11.** *For any  $\mathcal{Y}$ , the functor  $\mathbf{oblv}_{\mathcal{Y}}^r$  is conservative.*

*Proof.* The same as in the case of left crystals. □

Similarly, we have:

**Corollary 2.3.12.** *If  $\mathcal{Y}$  is classically formally smooth, then the functor*

$$\text{Crys}^r(\mathcal{Y}) \rightarrow \text{Tot}(\text{IndCoh}(\mathcal{Y}^\bullet / \mathcal{Y}_{\text{dR}}))$$

*is an equivalence.*

**2.4. Comparison of left and right crystals.** We remind the reader that we assume that all prestacks and DG schemes are locally almost of finite type.

2.4.1. Recall (see [GL:IndCoh, Sect. 8.4.3]) that for any  $S \in \mathrm{DGSch}_{\mathrm{aft}}$  there is a canonically defined functor

$$\Psi^\vee : \mathrm{QCoh}(S) \rightarrow \mathrm{IndCoh}(S),$$

given by tensoring with the dualizing sheaf  $\omega_S \in \mathrm{IndCoh}(S)$ , such that for  $f : S_1 \rightarrow S_2$ , the diagram

$$\begin{array}{ccc} \mathrm{QCoh}(S_1) & \xrightarrow{\Psi_{S_1}^\vee} & \mathrm{IndCoh}(S_1) \\ f^* \uparrow & & \uparrow f^! \\ \mathrm{QCoh}(S_2) & \xrightarrow{\Psi_{S_2}^\vee} & \mathrm{IndCoh}(S_2) \end{array}$$

canonically commutes. In fact, the above data upgrades to a natural transformation of functors

$$\mathrm{QCoh}_{\mathrm{DGSch}_{\mathrm{aft}}} \rightarrow \mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{aft}}}^!,$$

and hence gives rise to a functor

$$\Psi^\vee : \mathrm{QCoh}(\mathcal{Y}) \rightarrow \mathrm{IndCoh}(\mathcal{Y})$$

for any  $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ .

2.4.2. Applying  $\Psi^\vee$  to  $\mathcal{Y}_{\mathrm{dR}}$  for  $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ , we obtain a canonically defined functor

$$(2.4) \quad \Upsilon_{\mathcal{Y}} : \mathrm{Crys}^l(\mathcal{Y}) \rightarrow \mathrm{Crys}^r(\mathcal{Y}),$$

making the diagram

$$(2.5) \quad \begin{array}{ccc} \mathrm{Crys}^l(\mathcal{Y}) & \xrightarrow{\Upsilon_{\mathcal{Y}}} & \mathrm{Crys}^r(\mathcal{Y}) \\ \mathrm{oblv}_{\mathcal{Y}}^l \downarrow & & \downarrow \mathrm{oblv}_{\mathcal{Y}}^r \\ \mathrm{QCoh}_{\mathrm{lft}} & \xrightarrow{\Psi_{\mathcal{Y}}^\vee} & \mathrm{IndCoh}(\mathcal{Y}) \end{array}$$

commute.

In fact we obtain a natural transformation

$$\Upsilon : \mathrm{Crys}_{\mathrm{PreStk}_{\mathrm{lft}}}^l \rightarrow \mathrm{Crys}_{\mathrm{PreStk}_{\mathrm{lft}}}^r.$$

In particular, for  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  the diagram

$$\begin{array}{ccc} \mathrm{Crys}^l(\mathcal{Y}_1) & \xrightarrow{\Upsilon_{\mathcal{Y}_1}} & \mathrm{Crys}^r(\mathcal{Y}_1) \\ f^\dagger \uparrow & & \uparrow f^! \\ \mathrm{Crys}^l(\mathcal{Y}_2) & \xrightarrow{\Upsilon_{\mathcal{Y}_2}} & \mathrm{Crys}^r(\mathcal{Y}_2) \end{array}$$

commutes.

2.4.3. We claim:

**Proposition 2.4.4.** *The functor (2.4) is an equivalence.*

*Proof.* By Corollaries 2.2.4 and 2.3.9, the statement reduces to one saying that

$$\Upsilon_X : \mathrm{Crys}^l(X) \rightarrow \mathrm{Crys}^r(X)$$

is an equivalence for an affine DG scheme  $X$ .

Let  $i : X \hookrightarrow Z$  be a closed embedding, where  $Z$  is a smooth classical scheme, and let  $Y$  be the formal completion of  $Z$  along  $X$ . Since  $X_{\mathrm{dR}} \rightarrow Y_{\mathrm{dR}}$  is an isomorphism, the functors

$$f^\dagger : \mathrm{Crys}^l(Y) \rightarrow \mathrm{Crys}^l(X) \text{ and } f^! : \mathrm{Crys}^r(Y) \rightarrow \mathrm{Crys}^r(X)$$

are both equivalences. Hence, it is enough to prove the assertion for  $Y$ .

Let  $Y^\bullet/Y_{\text{dR}}$  be the Čech nerve of  $\text{PreStk}_{\text{laft}}$  corresponding to the map

$$p_{\text{dR},Y} : Y \rightarrow Y_{\text{dR}}.$$

Consider the commutative diagram

$$\begin{array}{ccc} \text{Crys}^l(Y) & \xrightarrow{\Upsilon_Y} & \text{Crys}^r(Y) \\ \downarrow & & \downarrow \\ \text{Tot}(\text{QCoh}(Y^\bullet/Y_{\text{dR}})) & \xrightarrow{\text{Tot}(\Psi_{Y^\bullet/Y_{\text{dR}}}^\vee)} & \text{Tot}(\text{IndCoh}(Y^\bullet/Y_{\text{dR}})). \end{array}$$

By Corollaries 2.1.7 and 2.3.12, the vertical arrows in the diagram are equivalences. Therefore, it suffices to show that for every  $i$ ,

$$\Psi_{Y^i/Y_{\text{dR}}}^\vee : \text{QCoh}(Y^i/Y_{\text{dR}}) \rightarrow \text{IndCoh}(Y^i/Y_{\text{dR}})$$

is an equivalence.

Recall (also from the proof of Proposition 1.3.2) that  $Y^i/Y_{\text{dR}}$  is the completion of the smooth classical scheme  $Z^i$  along the diagonal copy of  $X$ . Let us denote by  $U_i \subset Z^i$  the complementary open substack.

From [GL:IndSch, Propositions 7.1.3 and 7.4.5 and Diagram (7.14)], we obtain that we have a map of “short exact sequences” of DG categories

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{QCoh}(Y^i/Y_{\text{dR}}) & \longrightarrow & \text{QCoh}(Z^i) & \longrightarrow & \text{QCoh}(U_i) \longrightarrow 0 \\ & & \Psi_{Y^i/Y_{\text{dR}}}^\vee \downarrow & & \downarrow \Psi_{Z^i}^\vee & & \downarrow \Psi_{U_i}^\vee \\ 0 & \longrightarrow & \text{IndCoh}(Y^i/Y_{\text{dR}}) & \longrightarrow & \text{IndCoh}(Z^i) & \longrightarrow & \text{IndCoh}(U_i) \longrightarrow 0. \end{array}$$

Now, the functors

$$\Psi_{Z^i}^\vee : \text{QCoh}(Z^i) \rightarrow \text{IndCoh}(Z^i) \text{ and } \Psi_{U_i}^\vee : \text{QCoh}(U_i) \rightarrow \text{IndCoh}(U_i)$$

are both equivalences, since  $Z^i$  and  $U_i$  are smooth:

Indeed, by [GL:IndCoh, Proposition 8.4.4], for any  $S \in \text{DGSch}_{\text{aft}}$ , the functor  $\Psi_S^\vee$  is the dual of  $\Psi_S : \text{IndCoh}(S) \rightarrow \text{QCoh}(S)$ , and the latter is an equivalence if  $S$  is smooth by [GL:IndCoh, Lemma 1.1.6].

□

2.4.5. Proposition 2.4.4 allows us to identify left and right crystals for objects  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$ .

In other words, we can consider the category  $\text{Crys}(\mathcal{Y})$  equipped with two realizations: “left” and “right”, which incarnate themselves as forgetful functors  $\text{oblv}_{\mathcal{Y}}^l$  and  $\text{oblv}_{\mathcal{Y}}^r$  from  $\text{Crys}(\mathcal{Y})$  to  $\text{QCoh}(\mathcal{Y})$  and  $\text{IndCoh}(\mathcal{Y})$ , respectively.

The two forgetful functors are related by the commutative diagram

$$(2.6) \quad \begin{array}{ccc} & \text{Crys}(\mathcal{Y}) & \\ \text{oblv}_{\mathcal{Y}}^l \swarrow & & \searrow \text{oblv}_{\mathcal{Y}}^r \\ \text{QCoh}(\mathcal{Y}) & \xrightarrow{\Psi_{\mathcal{Y}}^\vee} & \text{IndCoh}(\mathcal{Y}) \end{array}$$

For a morphism  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  we have a naturally defined functor

$$f^! : \mathrm{Crys}(\mathcal{Y}_2) \rightarrow \mathrm{Crys}(\mathcal{Y}_1),$$

which makes the following diagrams commute

$$\begin{array}{ccc} \mathrm{Crys}(\mathcal{Y}_1) & \xleftarrow{f^!} & \mathrm{Crys}(\mathcal{Y}_2) \\ \mathrm{oblv}_{\mathcal{Y}_1}^l \downarrow & & \downarrow \mathrm{oblv}_{\mathcal{Y}_2}^l \\ \mathrm{QCoh}(\mathcal{Y}_1) & \xleftarrow{f^*} & \mathrm{QCoh}(\mathcal{Y}_2) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Crys}(\mathcal{Y}_1) & \xleftarrow{f^!} & \mathrm{Crys}(\mathcal{Y}_2) \\ \mathrm{oblv}_{\mathcal{Y}_1}^r \downarrow & & \downarrow \mathrm{oblv}_{\mathcal{Y}_2}^r \\ \mathrm{IndCoh}(\mathcal{Y}_1) & \xleftarrow{f^!} & \mathrm{IndCoh}(\mathcal{Y}_2). \end{array}$$

2.4.6. In the sequel, we shall use symbols  $\mathrm{Crys}(\mathcal{Y})$ ,  $\mathrm{Crys}^r(\mathcal{Y})$  and  $\mathrm{Crys}^l(\mathcal{Y})$  interchangeably with the former emphasizing that the statement is independent of realization (left or right) we choose, and the latter two, when a choice of the realization is important.

**2.5. Kashiwara's lemma.** A feature of the assignment  $\mathcal{Y} \mapsto \mathrm{Crys}(\mathcal{Y})$  is that Kashiwara's lemma becomes nearly tautological.

We will formulate and prove it for the incarnation of crystals as right crystals. By Proposition 2.4.4, this implies the corresponding assertion for left crystals. However, one could easily write the same proof in the language of left crystals instead.

2.5.1. Recall that a map  $i : \mathcal{X} \rightarrow \mathcal{Z}$  in  $\mathrm{PreStk}$  is called a closed embedding if it is such at the level of the underlying classical prestacks. I.e., if for every  $S \in (\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Z}}$  the base-changed map

$${}^{cl}(S \times_{\mathcal{Z}} \mathcal{X}) \rightarrow S$$

is a closed embedding; in particular,  ${}^{cl}(S \times_{\mathcal{Z}} \mathcal{X})$  is a classical affine scheme.

If  $\mathcal{X}, \mathcal{Z} \in \mathrm{PreStk}_{\mathrm{lft}}$ , it suffices to check the above condition for  $S \in (\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{Z}}$ .

2.5.2. For  $i : \mathcal{X} \hookrightarrow \mathcal{Z}$  a closed embedding of objects of  $\mathrm{PreStk}_{\mathrm{lft}}$ , let  $j : \mathring{\mathcal{Y}} \hookrightarrow \mathcal{Z}$  be the complementary open embedding. The induced map

$$j : \mathring{\mathcal{Y}}_{\mathrm{dR}} \rightarrow \mathcal{Z}_{\mathrm{dR}}$$

is also an open embedding. Consider the restriction functor

$$j_{\mathrm{dR}}^* := j^! : \mathrm{Crys}^r(\mathcal{Z}) \rightarrow \mathrm{Crys}^r(\mathring{\mathcal{Y}}).$$

It follows from [GL:IndCoh, Sect. 4.1], that the above functor admits a *fully faithful* right adjoint, denoted  $j_{\mathrm{dR},*}$ , such that for every  $S \in (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}}$  and

$$\mathring{S} := S \times_{\mathcal{Z}_{\mathrm{dR}}} \mathring{\mathcal{Y}}_{\mathrm{dR}} \xrightarrow{j_S} S,$$

the natural transformation in the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(S) & \longleftarrow & \mathrm{Crys}^r(\mathcal{Z}) \\ (j_S)_*^{\mathrm{IndCoh}} \uparrow & & \uparrow j_{\mathrm{dR},*} \\ \mathrm{IndCoh}(\overset{\circ}{S}) & \longleftarrow & \mathrm{Crys}^r(\overset{\circ}{\mathcal{Y}}) \end{array}$$

arising by adjunction, is an isomorphism.

In particular, the natural transformation

$$\mathbf{oblv}_{\mathcal{X}}^r \circ j_{\mathrm{dR},*} \rightarrow j_*^{\mathrm{IndCoh}} \circ \mathbf{oblv}_{\overset{\circ}{\mathcal{X}}}^r$$

is an isomorphism.

2.5.3. Let  $\mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}}$  denote the full subcategory of  $\mathrm{Crys}^r(\mathcal{Z})$  equal to  $\ker(j_{\mathrm{dR}}^*)$ .

Clearly, an object  $\mathcal{M} \in \mathrm{Crys}^r(\mathcal{Z})$  belongs to  $\mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}}$  if and only if for every object of  $(\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}}$ , i.e.,  $(S \in \mathrm{DGSch}_{\mathrm{aft}}, {}^{cl,red}S \rightarrow \mathcal{Z})$ , the corresponding object  $\mathcal{F}_S \in \mathrm{IndCoh}(S)$  lies in

$$\ker(j_S^{\mathrm{IndCoh},*}) : \mathrm{IndCoh}(S) \rightarrow \mathrm{IndCoh}(\overset{\circ}{S}),$$

where  $j_S^{\mathrm{IndCoh},*}$  is the same as  $j_S^!$ .

2.5.4. The embedding  $\mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}} \hookrightarrow \mathrm{Crys}^r(\mathcal{Z})$  admits a right adjoint, given by

$$\mathcal{M} \mapsto \mathrm{Cone}(\mathcal{M} \rightarrow j_{\mathrm{dR},*} \circ j_{\mathrm{dR}}^*(\mathcal{M}))[-1].$$

Hence, we can think of  $\mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}}$  as a co-localization of  $\mathrm{Crys}^r(\mathcal{Z})$ .

2.5.5. It is clear that the functor  $i^! : \mathrm{Crys}^r(\mathcal{Z}) \rightarrow \mathrm{Crys}^r(\mathcal{X})$  factors through the above co-localization functor

$$\mathrm{Crys}^r(\mathcal{Z}) \rightarrow \mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}} \xrightarrow{i^!} \mathrm{Crys}^r(\mathcal{X}).$$

Kashiwara's lemma says:

**Proposition 2.5.6.** *The above functor*

$$i^! : \mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}} \rightarrow \mathrm{Crys}^r(\mathcal{X})$$

*is an equivalence.*

*Proof.* Note that we have an isomorphism in  $\mathrm{PreStk}_{\mathrm{lft}}$ :

$$\mathrm{colim}_{S \in (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}}} S \times_{\mathcal{Z}_{\mathrm{dR}}} \mathcal{X}_{\mathrm{dR}} \simeq \mathcal{X}_{\mathrm{dR}}.$$

Furthermore,  $S^{\wedge} := S \times_{\mathcal{Z}_{\mathrm{dR}}} \mathcal{X}_{\mathrm{dR}}$  identifies with the formal completion of  $S$  along

$${}^{red,cl}S \times_{{}^{cl,red}\mathcal{Z}} {}^{cl,red}\mathcal{X}.$$

Hence, the category  $\mathrm{Crys}^r(\mathcal{Y})$  can be described as

$$\lim_{S \in ((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}})^{op}} \mathrm{IndCoh}(S^{\wedge}).$$

By definition, the category  $\mathrm{Crys}^r(\mathcal{Z})_{\mathcal{X}}$  is given by

$$\lim_{S \in ((\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{Z}_{\mathrm{dR}}})^{op}} \ker \left( \mathrm{IndCoh}(S) \rightarrow \mathrm{IndCoh}(\overset{\circ}{S}) \right).$$

Now, [GL:IndSch, Proposition 7.4.5] says that for any  $S$  as above,  $!$ -pullback gives an equivalence

$$\ker \left( \mathrm{IndCoh}(S) \rightarrow \mathrm{IndCoh}(\overset{\circ}{S}) \right) \rightarrow \mathrm{IndCoh}(S^\wedge),$$

as desired.  $\square$

*Remark 2.5.7.* If we phrased the above proof in terms of left crystals instead of right crystals, we would have used Proposition 7.1.3 of [GL:IndSch] instead of Proposition 7.4.5.

### 3. DESCENT PROPERTIES OF CRYSTALS

In this section all prestacks, including DG schemes and DG indschemes are assumed locally almost of finite type, unless explicitly stated otherwise.

The goal of this section is to establish a number of properties concerning the behavior of crystals on DG schemes and DG indschemes.

#### 3.1. Descent for crystals.

3.1.1. Consider the fppf topology on the category  $\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}$ , induced from the fppf topology on  $\mathrm{DGSch}^{\mathrm{aff}}$  (see [GL:Stacks, Sect. 2.2]).

Consider the functor

$$\mathrm{Crys}_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}}^r := \mathrm{Crys}_{\mathrm{PreStk}_{\mathrm{lft}}}^r \big|_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}} : (\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}})^{op} \rightarrow \mathrm{DGCat}.$$

We will prove:

**Proposition 3.1.2.** *The functor  $\mathrm{Crys}_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}}^r$  satisfies fppf descent.*

3.1.3. The fppf topology on  $\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}$  induces the fppf topology on the full subcategory  $<^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}} \subset \mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}$ .

Thus, Proposition 3.1.2 gives:

**Corollary 3.1.4.** *The functor*

$$\mathrm{Crys}_{<^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}}^r := \mathrm{Crys}_{\mathrm{PreStk}_{\mathrm{lft}}}^r \big|_{<^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}} : (<^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}})^{op} \rightarrow \mathrm{DGCat}$$

*on  $<^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}$  satisfies fppf descent.*

Thus, formally, we obtain:

**Corollary 3.1.5.** *Let  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  be a map in  $\mathrm{PreStk}_{\mathrm{lft}}$  which is a surjection in the fppf topology. Then the natural map*

$$\mathrm{Crys}(\mathcal{Y}_2) \rightarrow \mathrm{Tot}(\mathrm{Crys}(\mathcal{Y}_1^\bullet / \mathcal{Y}_2))$$

*is an equivalence.*



3.1.6. *Proof of Proposition 3.1.2.* As in [GL:IndCoh, Theorem 7.3.2], fpppf descent is a combination of Nisnevich descent and finite-flat descent<sup>3</sup>. We shall show that  $\text{Crys}_{\text{DGSch}_{\text{aft}}^{\text{aff}}}^r$  satisfies étale descent and proper-surjective descent.

The étale descent statement is clear: if  $S' \rightarrow S$  is an étale cover in  $\text{DGSch}_{\text{aft}}^{\text{aff}}$  then the corresponding map  $S'_{\text{dR}} \rightarrow S_{\text{dR}}$  is a schematic, étale and surjective map in  $\text{PreStk}_{\text{lft}}$ . In particular, it is a cover for the fpppf topology, and the statement follows from the fpppf descent for IndCoh, see [GL:IndCoh, Corollary 9.4.5].

If  $f : S' \rightarrow S$  is a proper surjective map, then the corresponding map  $f_{\text{dR}} : S'_{\text{dR}} \rightarrow S_{\text{dR}}$  has the property that its base change by any DG scheme  $S_1 \rightarrow S_{\text{dR}}$  yields a proper and surjective map  $S'_1 := S_1 \times_{S_{\text{dR}}} S'_{\text{dR}} \rightarrow S_1$ . Hence, the assertion follows from [GL:IndSch, Lemma 2.6.3].  $\square$

3.2. **The infinitesimal groupoid.** In this subsection, let  $\mathcal{X}$  be an object on  $\text{DGindSch}_{\text{lft}}$ .

3.2.1. Let  $\mathcal{X}^\bullet/\mathcal{X}_{\text{dR}}$  denote the simplicial object of Sect. 1.2.2. As was remarked already, each  $\mathcal{X}^i/\mathcal{X}_{\text{dR}}$  is the formal completion of  $\mathcal{X}^i$  along the main diagonal. In particular, all  $\mathcal{X}^i/\mathcal{X}_{\text{dR}}$  also belong to  $\text{DGindSch}_{\text{lft}}$ .

We shall refer to

$$\mathcal{X} \times_{\mathcal{X}_{\text{dR}}} \mathcal{X} \rightrightarrows \mathcal{X}$$

as the infinitesimal groupoid of  $\mathcal{X}$ .

3.2.2. Consider the cosimplicial category  $\text{IndCoh}(\mathcal{X}^\bullet/\mathcal{X}_{\text{dR}})$ .

**Proposition 3.2.3.** *The functor*

$$\text{Crys}^r(\mathcal{X}) \rightarrow \text{Tot}(\text{IndCoh}(\mathcal{X}^\bullet/\mathcal{X}_{\text{dR}})),$$

*defined by the augmentation, is an equivalence.*

*Remark 3.2.4.* Note that by Corollary 2.3.12, the assertion of the proposition holds also for  $\mathcal{X}$  replaced any classically formally smooth object  $\mathcal{Y} \in \text{PreStk}_{\text{lft}}$ .

*Proof.* It suffices to show that for any  $S \in \text{DGSch}_{\text{aft}}$  and a map  $S \rightarrow \mathcal{X}_{\text{dR}}$ , the functor

$$\text{IndCoh}(S) \rightarrow \text{Tot} \left( \text{IndCoh} \left( S \times_{\mathcal{X}_{\text{dR}}} \mathcal{X}^\bullet / \mathcal{X}_{\text{dR}} \right) \right)$$

is an equivalence.

Note that the simplicial object  $S \times_{\mathcal{X}_{\text{dR}}} (\mathcal{X}^\bullet/\mathcal{X}_{\text{dR}})$  is the Čech nerve corresponding to

$$S \times_{\mathcal{X}_{\text{dR}}} \mathcal{X} \rightarrow S,$$

while  $S \times_{\mathcal{X}_{\text{dR}}} \mathcal{X} \in \text{DGindSch}_{\text{lft}}$  and its map to  $S$  is proper (in fact, ind-finite). Indeed,

$$S \times_{\mathcal{X}_{\text{dR}}} \mathcal{X}$$

identifies with the formal completion of  $S \times \mathcal{X}$  along the graph of  $^{cl, red} S \rightarrow \mathcal{X}$ .

Hence, our assertion follows from [GL:IndSch, Lemma 2.6.3].  $\square$

---

<sup>3</sup>This observation was explained to us by J. Lurie.

3.2.5. As an immediate corollary of Proposition 3.2.3, we obtain:

**Corollary 3.2.6.** *The forgetful functor*

$$\mathbf{oblv}_{\mathcal{X}}^r : \mathrm{Crys}(\mathcal{X}) \rightarrow \mathrm{IndCoh}(\mathcal{X})$$

*admits a left adjoint.*

*Proof.* This follows from [GL:IndSch, Sect. 2.6.1]. □

We shall denote the left adjoint to  $\mathbf{oblv}_{\mathcal{X}}^r$  by  $\mathbf{ind}_{\mathcal{X}}^r$ .

3.2.7. By the Barr-Beck-Lurie theorem, we obtain that the endo-functor

$$\mathbf{oblv}_{\mathcal{X}}^r \circ \mathbf{ind}_{\mathcal{X}}^r$$

of  $\mathrm{IndCoh}(\mathcal{X})$  has a structure of monad, and  $\mathrm{Crys}^r(\mathcal{X})$  identifies with the category of modules over this monad. From [GL:IndSch, Sect. 2.6.1], we obtain:

**Corollary 3.2.8.** *We have a canonical isomorphism of functors*

$$\mathbf{oblv}_{\mathcal{X}}^r \circ \mathbf{ind}_{\mathcal{X}}^r \simeq (p_2)_*^{\mathrm{IndCoh}} \circ (p_1)^!,$$

where  $p_1, p_2$  are the two projections

$$\mathcal{X} \times_{\mathcal{X}_{\mathrm{dR}}} \mathcal{X} \rightrightarrows \mathcal{X}.$$

**3.3. The induction functor and infinitesimal groupoid for left crystals.** It follows from Lemma 1.2.4 that for a smooth classical scheme  $X$ , the analogue of Proposition 3.2.3 holds for left crystals, i.e., the functor

$$(3.1) \quad \mathrm{Crys}^l(X) = \mathrm{QCoh}(X_{\mathrm{dR}}) \rightarrow \mathrm{Tot}(\mathrm{QCoh}(X^\bullet/X_{\mathrm{dR}}))$$

is an equivalence.

By Proposition 3.2.3, the analogous statement for right crystals is true for any DG scheme  $X$  (and even a DG indscheme). However, this is not the case for left crystals.

3.3.1. We claim:

**Proposition 3.3.2.** *If a DG scheme  $X$  is eventually coconnective, then the functor (3.1) is an equivalence.*

*Remark 3.3.3.* The statement of the proposition is false without the eventual coconnectivity assumption. For instance, consider the DG scheme  $X = \mathrm{Spec}(k[\alpha])$ , where  $\alpha$  is in degree -2. One can show that quasi-coherent sheaves on the infinitesimal groupoid of  $X$  are given by modules over the non-commutative ring  $k\{\alpha, \partial_\alpha\}/[\partial_\alpha, \alpha] = 1$ , where  $\partial_\alpha$  is in degree 2. But, by definition, the category of crystals over  $X$  is given by modules over  $k$ . Nevertheless, by Corollary 2.1.7, the proposition is still true for any prestack which is classically formally smooth.

*Proof.* We have a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Crys}^l(X) & \longrightarrow & \mathrm{Tot}(\mathrm{QCoh}(X^\bullet/X_{\mathrm{dR}})) \\ \Upsilon_X \downarrow & & \downarrow \mathrm{Tot}(\Psi_{X^\bullet/X_{\mathrm{dR}}}^\vee) \\ \mathrm{Crys}^r(X) & \longrightarrow & \mathrm{Tot}(\mathrm{IndCoh}(X^\bullet/X_{\mathrm{dR}})). \end{array}$$

with the left vertical map and the lower horizontal map being equivalences. Hence, we obtain that  $\mathrm{Crys}^l(X)$  is a retract of  $\mathrm{Tot}(\mathrm{QCoh}(X^\bullet/X_{\mathrm{dR}}))$ .

Recall that if  $Z$  is an eventually coconnective DG scheme, the functor

$$\Psi_Z^\vee : \mathrm{QCoh}(Z) \rightarrow \mathrm{IndCoh}(Z)$$

is fully faithful (see [GL:IndCoh, Corollary 8.5.1]. Hence, by [GL:IndSch, Propositions 7.1.3 and 7.4.5], the same is true for the completion of an eventually coconnective DG scheme along a Zariski closed subset. Hence, the functors

$$\Psi_{X^i/X_{\mathrm{dR}}}^\vee : \mathrm{QCoh}(X^i/X_{\mathrm{dR}}) \rightarrow \mathrm{IndCoh}(X^i/X_{\mathrm{dR}})$$

are fully faithful. Thus, the functor  $\mathrm{Tot}(\Psi_{X^\bullet/X_{\mathrm{dR}}}^\vee)$  in the above commutative diagram is also fully faithful. But it is also essentially surjective since the identity functor is its retract.  $\square$

3.3.4. For a DG indscheme  $\mathcal{X}$ , define the functor

$$\mathbf{ind}_X^l : \mathrm{QCoh}(\mathcal{X}) \rightarrow \mathrm{Crys}(X)$$

as

$$\mathbf{ind}_X^l := \mathbf{ind}_X^r \circ \Psi_{\mathcal{X}}^\vee.$$

We claim:

**Lemma 3.3.5.** *If  $X$  is an eventually coconnective DG scheme, the functors  $(\mathbf{ind}_X^l, \mathbf{oblv}_X^l)$  are mutually adjoint.*

*Remark 3.3.6.* The assertion of the lemma would be false if we dropped the assumption that  $X$  be eventually coconnective.

*Proof.* Recall (see [GL:IndCoh], Sect. 8.5.3) that for  $X$  eventually coconnective, the functor  $\Psi_X^\vee$  admits a right adjoint, denoted  $\Xi_X^\vee$ ; moreover, the functor  $\Psi_X^\vee$  itself is fully faithful.

We obtain that the right adjoint to  $\mathbf{ind}_X^l$  is given by

$$\Xi_X^\vee \circ \mathbf{oblv}_X^r \simeq \Xi_X^\vee \circ \Psi_X^\vee \circ \mathbf{oblv}_X^l \simeq \mathbf{oblv}_X^l,$$

as required.  $\square$

### 3.4. The sheaf of differential operators.

3.4.1. Recall that for  $\mathcal{X} \in \mathrm{DGindSch}$ , the category  $\mathrm{IndCoh}(\mathcal{X})$  is dualizable and canonically self-dual. Hence, for  $\mathcal{X}, \mathcal{Y} \in \mathrm{DGindSch}$ , the category  $\mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{Y}))$  identifies with

$$\mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{Y}) \simeq \mathrm{IndCoh}(\mathcal{X} \times \mathcal{Y}).$$

Expilcilty, an object  $K \in \mathrm{IndCoh}(\mathcal{X} \times \mathcal{Y})$  defines a functor  $\mathbf{F}_K : \mathrm{IndCoh}(\mathcal{X}) \rightarrow \mathrm{IndCoh}(\mathcal{Y})$  by

$$\mathcal{F} \mapsto (p_2)_*^{\mathrm{IndCoh}} \circ (\Delta_{\mathcal{X}} \times \mathrm{id}_{\mathcal{Y}})^!(\mathcal{F} \boxtimes K),$$

where  $p_2 : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$  is the projection map.

In particular, the endo-functor  $\mathbf{oblv}_{\mathcal{X}}^r \circ \mathbf{ind}_{\mathcal{X}}^r$  defines an object, denoted

$$\mathcal{D}_{\mathcal{X}}^r \in \mathrm{IndCoh}(\mathcal{X} \times \mathcal{X}).$$

We will identify this object.

3.4.2. Let  $\Delta_{\mathcal{X}}$  denote the diagonal map  $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ , and let  $\widehat{\Delta}_{\mathcal{X}}$  denote the map

$$\mathcal{X} \times_{\mathcal{X}_{\mathrm{dR}}} \mathcal{X} \simeq (\mathcal{X} \times \mathcal{X})_{\mathcal{X}}^{\wedge} \rightarrow \mathcal{X} \times \mathcal{X}.$$

**Proposition 3.4.3.** *There is a canonical isomorphism in  $\mathrm{IndCoh}(\mathcal{X} \times \mathcal{X})$*

$$\mathcal{D}_{\mathcal{X}}^r \simeq (\widehat{\Delta}_{\mathcal{X}})_*^{\mathrm{IndCoh}}(\omega_{\mathcal{X} \times_{\mathcal{X}_{\mathrm{dR}}} \mathcal{X}}).$$

*Proof.* We begin with the following general observation.

Suppose that we have a functor  $F \in \mathrm{Func}_{\mathrm{cont}}(\mathrm{IndCoh}(\mathcal{X}), \mathrm{IndCoh}(\mathcal{Y}))$  given by a correspondence, i.e. we have a diagram

$$\begin{array}{ccc} & \mathcal{Z} & \\ p_1 \swarrow & & \searrow p_2 \\ \mathcal{X} & & \mathcal{Y} \end{array}$$

of DG indschemes, and  $F := (p_2)_*^{\mathrm{IndCoh}} \circ p_1^!$ . Let

$$i : \mathcal{Z} \rightarrow \mathcal{X} \times \mathcal{Y}$$

be the induced product map.

**Lemma 3.4.4.** *In the above situation, the functor*

$$(p_2)_*^{\mathrm{IndCoh}} \circ p_1^! : \mathrm{IndCoh}(\mathcal{X}) \rightarrow \mathrm{IndCoh}(\mathcal{Y})$$

*is given by the kernel  $K = i_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Z}})$ .*

*Proof.* We have a Cartesian square

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{p_1 \times \mathrm{id}_{\mathcal{Z}}} & \mathcal{X} \times \mathcal{Z} \\ i \downarrow & & \downarrow \mathrm{id}_{\mathcal{X}} \times i \\ \mathcal{X} \times \mathcal{Y} & \xrightarrow{\Delta_{\mathcal{X}} \times \mathrm{id}_{\mathcal{Y}}} & \mathcal{X} \times \mathcal{X} \times \mathcal{Y}. \end{array}$$

For  $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ , we have

$$(p_2)_*^{\mathrm{IndCoh}} \circ p_1^!(\mathcal{F}) \simeq (p_2)_*^{\mathrm{IndCoh}} \circ i_*^{\mathrm{IndCoh}} \circ (p_1 \times \mathrm{id}_{\mathcal{Z}})^!(\mathcal{F} \boxtimes \omega_{\mathcal{Z}})$$

where  $p_2 : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$  is the projection map. But by [GL:IndSch, Proposition 2.5.14],<sup>4</sup>

$$i_*^{\mathrm{IndCoh}} \circ (p_1 \times \mathrm{id}_{\mathcal{Z}})^!(\mathcal{F} \boxtimes \omega_{\mathcal{Z}}) \simeq (\Delta_{\mathcal{X}} \times \mathrm{id}_{\mathcal{Y}})^!(\mathcal{F} \boxtimes i_*^{\mathrm{IndCoh}}(\omega_{\mathcal{Z}})).$$

□

We apply this lemma to prove Proposition 3.4.3 as follows:

By Corollary 3.2.8, we have that the functor  $\mathbf{oblv}_{\mathcal{X}}^r \circ \mathbf{ind}_{\mathcal{X}}^r$  is given by the correspondence

$$\begin{array}{ccc} & (\mathcal{X} \times \mathcal{X})_{\mathcal{X}}^{\wedge} & \\ p_1 \swarrow & & \searrow p_2 \\ \mathcal{X} & & \mathcal{X}. \end{array}$$

The assertion now follows from Lemma 3.4.4.

□

<sup>4</sup>Strictly speaking, the base change isomorphism was stated in [GL:IndSch, Proposition 2.5.14] only in the case when the vertical arrow is proper, which translates into  $i$  being proper. For the proof of Proposition 3.4.3 we will apply it in such a situation.

3.4.5. Let  $X$  be a quasi-compact eventually coconnective DG scheme. The pair of adjoint functors

$$\mathbf{ind}_X^l : \mathrm{QCoh}(X) \rightleftarrows \mathrm{Crys}(X) : \mathbf{oblv}_X^l$$

defines a monad  $\mathbf{oblv}_X^l \circ \mathbf{ind}_X^l$  acting on  $\mathrm{QCoh}(X)$ .

Recall now that if

$$\mathbf{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathbf{G}$$

is a pair of adjoint functors, and  $\mathcal{M}_{\mathcal{D}}$  is a monad acting on  $\mathcal{D}$ , then the composition

$$\mathbf{G} \circ \mathcal{M}_{\mathcal{D}} \circ \mathbf{F}$$

also has a natural structure of monad. In fact, it corresponds to the composition of the functor

$$\mathcal{C} \xleftarrow{\mathbf{G}} \mathcal{D} \xleftarrow{\mathbf{oblv}^{\mathcal{M}}} \mathcal{M}\text{-mod}(\mathcal{D})$$

with its left adjoint.

We deduce that the monad  $\mathbf{oblv}_X^l \circ \mathbf{ind}_X^l$  is obtained from the monad  $\mathbf{oblv}_X^r \circ \mathbf{ind}_X^r$  via the pair of adjoint functors

$$\Psi_X^\vee : \mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}(X) : \Xi_X^\vee.$$

3.4.6. Recall now that the category  $\mathrm{QCoh}(X)$  is also compactly generated and self-dual. Under the identifications

$$\mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X) \text{ and } \mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X),$$

the dual of the functor  $\Xi_X$  is  $\Xi_X^\vee$ , and the dual of  $\Psi_X$  is  $\Psi_X^\vee$  (hence, the notation), see [GL:IndCoh], Proposition 8.4.4 and Sect. 8.5.

Let  $\mathcal{D}_X^l$  denote the object of

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X \times X)$$

corresponding to the functor  $\mathbf{oblv}_X^l \circ \mathbf{ind}_X^l$ . From the above, we obtain:

**Lemma 3.4.7.** *The objects*

$$\mathcal{D}_X^l \in \mathrm{QCoh}(X) \otimes \mathrm{QCoh}(X) \text{ and } \mathcal{D}_X^r \in \mathrm{IndCoh}(X) \otimes \mathrm{IndCoh}(X)$$

*are related by*

$$\mathcal{D}_X^l \simeq (\Psi_X \otimes \Xi_X^\vee)(\mathcal{D}_X^r).$$

*Remark 3.4.8.* Assume that  $X$  is a smooth classical scheme. In this case  $\mathcal{D}_X^l$  belongs to the heart of the t-structure on  $\mathrm{QCoh}(X \times X)$ . We shall see in Sect. 4.7.3 that  $\mathcal{D}_X^l$  identifies with the usual sheaf of differential operators on  $X$ .

Moreover, in this case the functors  $(\Xi_X, \Psi_X)$ , and hence also  $(\Psi_X^\vee, \Xi_X^\vee)$  are equivalences. In this case, from Lemma 3.4.7, we obtain that as an  $\mathcal{O}_X$ -bimodule,

$$\mathcal{D}_X^r \simeq \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X^l.$$

#### 4. t-STRUCTURES ON CRYSTALS

The category of crystals has two natural t-structures, which are compatible with the left and right realizations respectively. One of the main advantages of the right realization is that the t-structure compatible with it is much better behaved. In this section, we will define the two t-structures and prove some of their basic properties.

**4.1. The left t-structure.** In this subsection, we do not make the assumption that prestacks be locally almost of finite type.

4.1.1. Recall [GL:QCoh, Sec. 1.2.3] that for any prestack  $\mathcal{Z}$ , the category  $\mathrm{QCoh}(\mathcal{Z})$  has a canonical t-structure characterized by the following condition: an object  $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Z})$  belongs to  $\mathrm{QCoh}(\mathcal{Z})^{\leq 0}$  if and only if for every  $S \in \mathrm{DGSch}^{\mathrm{aff}}$  and a map  $\phi : S \rightarrow \mathcal{Z}$ , we have

$$\phi^*(\mathcal{F}) \in \mathrm{QCoh}(S)^{\leq 0}.$$

In particular, taking  $\mathcal{Z} = \mathcal{Y}_{\mathrm{dR}}$  for some prestack  $\mathcal{Y}$ , we obtain a canonical t-structure on  $\mathrm{Crys}^l(\mathcal{Y})$ , which we shall call the “left t-structure.”

By definition, the functor

$$\mathbf{oblv}^l : \mathrm{Crys}^l(\mathcal{X}) \rightarrow \mathrm{QCoh}(\mathcal{X})$$

is right t-exact for the left t-structure.

4.1.2. In general, the left t-structure is quite poorly behaved. However, we have the following assertion:

**Proposition 4.1.3.** *Let  $\mathcal{Y}$  be a classically formally smooth prestack. Then*

$$\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})^{\leq 0} \Leftrightarrow \mathbf{oblv}_{\mathcal{Y}}^l(\mathcal{M}) \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}.$$

*Proof.* We need to show that if  $\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})$  is such that  $\mathbf{oblv}_{\mathcal{X}}^l(\mathcal{M}) \in \mathrm{QCoh}(\mathcal{Y})^{\leq 0}$  then  $\mathcal{M} \in \mathrm{Crys}^l(\mathcal{Y})^{\leq 0}$ . I.e., we need to show that for every  $S \in \mathrm{DGSch}^{\mathrm{aff}}$  and  $\phi : S \rightarrow \mathcal{Y}_{\mathrm{dR}}$ ,  $\phi^*(\mathcal{M}) \in \mathrm{QCoh}(S)^{\leq 0}$ .

Let  $\mathcal{Y}^\bullet / \mathcal{Y}_{\mathrm{dR}}$  be the Čech nerve of the map  $p_{\mathrm{dR}, \mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y}_{\mathrm{dR}}$ . By Lemma 1.2.4, there exists a map  $\phi' : S \rightarrow \mathcal{Y}$  and an isomorphism  $\phi \simeq p_{\mathrm{dR}, \mathcal{Y}} \circ \phi'$ . The assertion now follows from the fact that  $\phi'^*$  is right t-exact.  $\square$

4.2. **The right t-structure.** From this point until the end of this section we reinstate the assumption that all prestacks are locally almost of finite type, unless explicitly stated otherwise.

4.2.1. Let  $X$  be a DG scheme (or an Artin stack). Recall that the category  $\mathrm{IndCoh}(X)$  has a natural t-structure, compatible with filtered colimits, see [GL:IndCoh, Sect. 1.2] (in the case of quasi-compact schemes) and [GL:IndCoh, Sect. 10.2.5] (for Artin stacks).

It is characterized by the property that an object  $\mathcal{M} \in \mathrm{IndCoh}(X)$  is connective (i.e., lies in  $\mathrm{IndCoh}(X)^{\geq 0}$ ) if and only if its image under the functor  $\Psi_X : \mathrm{IndCoh}(X) \rightarrow \mathrm{QCoh}(X)$  is connective.

4.2.2. We define the right t-structure on  $\mathrm{Crys}^r(X)$  by declaring that

$$\mathcal{M} \in \mathrm{Crys}^r(X)^{\geq 0} \Leftrightarrow \mathbf{oblv}_X^r(\mathcal{M}) \in \mathrm{IndCoh}(X)^{\geq 0}.$$

In what follows, we shall refer to the right t-structure on  $\mathrm{Crys}^r(X)$  as “the” t-structure on crystals. In other words, by default the t-structure we shall consider will be the right one. By construction, this t-structure is also compatible with filtered colimits, since  $\mathbf{oblv}_X^r$  is continuous.

For the rest of this section, we shall specialize to the case of DG schemes.

4.2.3. As in [GL:IndCoh, Corollary 4.2.4] one shows that this t-structure in Zariski-local, i.e., an object is connective/coconnective if and only if its restriction to a Zariski cover has this property.

4.2.4. By construction, the forgetful functor  $\mathbf{oblv}_X^r$  is left t-exact. Hence, by adjunction, the functor  $\mathbf{ind}_X^r$  is right t-exact.

We now claim:

**Proposition 4.2.5.**

- (a) *The functor  $\mathbf{ind}_X^r$  is t-exact.*
- (b) *If  $X$  is a smooth classical scheme, then  $\mathbf{oblv}_X^r$  is t-exact.*
- (c) *For a quasi-compact scheme  $X$ , the functor  $\mathbf{oblv}_X^r$  is of bounded cohomological amplitude.*

4.2.6. Before we prove the above proposition, we need to establish the following:

Let  $i : X \rightarrow Z$  be a closed embedding of DG schemes. Let  $i_{dR,*}$  denote the functor  $\mathbf{Crys}^r(X) \rightarrow \mathbf{Crys}^r(Z)$  equal to the composition

$$\mathbf{Crys}^r(X) \xrightarrow{('i^!)^{-1}} \mathbf{Crys}^r(Z)_X \hookrightarrow \mathbf{Crys}^r(Z),$$

which, by construction, is the left adjoint of  $i^!$ .

We have:

**Proposition 4.2.7.** *The functor  $i_{dR,*} : \mathbf{Crys}^r(X) \rightarrow \mathbf{Crys}^r(Z)$  is t-exact.*

*Proof.* The fact that  $i_{dR,*}$  is right t-exact is formal:

Note that the functor  $i^! : \mathbf{IndCoh}(Z) \rightarrow \mathbf{IndCoh}(X)$  is left t-exact, because it is the right adjoint of  $i_*^{\mathbf{IndCoh}}$ , which is t-exact, since  $i$  is a closed embedding. By definition, this implies that the functor

$$i^! : \mathbf{Crys}^r(Z) \rightarrow \mathbf{Crys}^r(X)$$

is left t-exact. Hence, the functor  $i_{dR,*}$  is right t-exact, being the left adjoint of a left t-exact functor.

It remains to show that  $i_{dR,*}$  is left t-exact, which is equivalent to  $\mathbf{oblv}_Z^r \circ i_{dR,*}$  being left t-exact. Recall the description of the latter functor given in the course of the proof of Proposition 2.5.6. We need to show that if  $\mathcal{F} \in \mathbf{IndCoh}(Z)_X \subset \mathbf{IndCoh}(Z)$  is an object such that

$$i^!(\mathcal{F}) \in \mathbf{IndCoh}(X)^{>0}$$

then  $\mathcal{F} \in \mathbf{IndCoh}(Z)^{>0}$ .

By [GL:IndSch, Proposition 7.4.5 and Sect. 7.4.3], we have

$$\mathcal{F} \simeq \operatorname{colim}_{X_1} (i_1)_*^{\mathbf{IndCoh}} (i_1^!(\mathcal{F})),$$

where the colimit is taken over the (filtered) family of closed embeddings

$$X \xrightarrow{i_1} X_1 \xrightarrow{i_1} Z,$$

such that  ${}^{cl,red}X \rightarrow {}^{cl,red}X_1$  is an isomorphism.

Hence, it suffices to show the following: if  $\mathcal{F} \in \mathbf{IndCoh}(X_1)$  is such that

$$'i_1^!(\mathcal{F}) \in \mathbf{IndCoh}(X)^{>0},$$

then  $\mathcal{F} \in \mathbf{IndCoh}(X_1)^{>0}$ .

Recall that the category  $\mathbf{IndCoh}(X_1)^{>0}$  is, by definition, the right orthogonal of  $\mathbf{Coh}(X_1)^{\leq 0}$ . Since  $'i_1 : X \rightarrow X_1$  is a nil-embedding, this is the same as the right orthogonal of the essential image of  $\mathbf{Coh}(X)^{\leq 0}$  under  $('i_1)_*^{\mathbf{IndCoh}}$ . I.e.,  $\mathcal{F} \in \mathbf{IndCoh}(X_1)^{>0}$  if and only if  $'i_1^!(\mathcal{F})$  is right orthogonal to  $\mathbf{Coh}(X)^{\leq 0}$ , i.e., that  $'i_1^!(\mathcal{F}) \in \mathbf{IndCoh}(X)^{>0}$ .

□

**Corollary 4.2.8.** *If a map  $X_1 \rightarrow X_2$  of DG schemes induces an isomorphism*

$$cl, red X_1 \rightarrow cl, red X_2,$$

*then the corresponding t-structures on  $\mathrm{Crys}^r(X_1) \simeq \mathrm{Crys}^r(X_2)$  coincide.*

4.2.9. *Proof of Proposition 4.2.5.* First, suppose that  $X$  is a smooth classical scheme. In particular, the functor  $\Psi_X$  of [GL:IndCoh, Sect. 1.1.5] defines an equivalence  $\mathrm{IndCoh}(X) \rightarrow \mathrm{QCoh}(X)$ .

It follows from Corollary 3.2.8 that in this case the monad  $\mathbf{oblv}_X^r \circ \mathbf{ind}_X^r$  is t-exact. By definition, this implies that the functor  $\mathbf{ind}_X^r$  is left t-exact. Hence, since it was right t-exact a priori, it is t-exact.

By the definition of the t-structure on  $\mathrm{Crys}^r(X)$ , the essential image of  $\mathrm{IndCoh}(X)^{\leq 0}$  under  $\mathbf{ind}_X^r$  generates  $\mathrm{Crys}^r(X)^{\leq 0}$  by taking colimits. Hence, in order to show that  $\mathbf{oblv}_X^r$  is right t-exact (in the smooth case), it suffices to show that  $\mathbf{oblv}_X^r \circ \mathbf{ind}_X^r$  sends  $\mathrm{IndCoh}(X)^{\leq 0}$  to itself. But the latter follows from the t-exactness of  $\mathbf{oblv}_X^r \circ \mathbf{ind}_X^r$ .

Let us now show that  $\mathbf{ind}_X^r$  is t-exact for any DG scheme. By Sect. 4.2.3, the question is Zariski-local, so we can assume without loss of generality that  $X$  is affine. Now, let  $i : X \rightarrow Z$  be an embedding of  $X$  into a smooth classical scheme. Since the functor  $i_{dR,*}$  is t-exact (by Proposition 4.2.7) and conservative (by Proposition 2.5.6), it suffices to show that the functor  $i_{dR,*} \circ \mathbf{ind}_X^r$  is t-exact.

However, by adjunction,

$$\mathbf{ind}_Z^r \circ i_*^{\mathrm{IndCoh}} \simeq i_{dR,*} \circ \mathbf{ind}_X^r,$$

and the left-hand side is t-exact because  $i_*^{\mathrm{IndCoh}}$  is t-exact for a closed embedding.

Finally, let us show that  $\mathbf{oblv}_X^r$  is of bounded cohomological amplitude for a quasi-compact  $X$ . The question readily reduces to the case when  $X$  is affine, and let  $i : X \hookrightarrow Z$  be as above. It suffices to show that the functor  $i^! : \mathrm{IndCoh}(Z) \rightarrow \mathrm{IndCoh}(X)$  is of bounded cohomological amplitude, but the latter follows easily from the fact that  $Z$  is regular.

□

**4.3. Right t-structure on crystals on indschemes.** Let  $\mathcal{X}$  be a DG indscheme. Fix a presentation of  $\mathcal{X}$

$$(4.1) \quad \mathcal{X} = \operatorname{colim}_{\alpha} X_{\alpha}$$

as in [GL:IndSch, Prop. 1.3.2]. For each  $\alpha$ , let  $i_{\alpha}$  denote the corresponding closed embedding  $X_{\alpha} \rightarrow \mathcal{X}$ . As in Sect. 4.2.6, we have adjoint functors

$$(i_{\alpha})_{dR,*} : \mathrm{Crys}^r(X) \rightleftarrows \mathrm{Crys}^r(\mathcal{X}) : i_{\alpha}^!$$

Furthermore, by [GL:DG, Sect. 1.3.3], we have that

$$\mathrm{Crys}^r(\mathcal{X}) \simeq \operatorname{colim}_{\alpha} \mathrm{Crys}^r(X_{\alpha}),$$

where for  $\alpha_2 \geq \alpha_1$ , the functor  $\mathrm{Crys}^r(X_{\alpha_2}) \rightarrow \mathrm{Crys}^r(X_{\alpha_1})$  is given by  $(i_{\alpha_1, \alpha_2})_{dR,*}$ .



4.3.1. Recall from [GL:IndSch, Sect. 2.4.9] that  $\mathrm{IndCoh}(\mathcal{X})$  has a natural t-structure compatible with filtered colimits.

Using this t-structure on  $\mathrm{IndCoh}(\mathcal{X})$ , we can define the right t-structure on  $\mathrm{Crys}^r(\mathcal{X})$ . Namely, we have

$$\mathcal{M} \in \mathrm{Crys}^r(\mathcal{X})^{\geq 0} \Leftrightarrow \mathbf{oblv}_X^r(\mathcal{M}) \in \mathrm{IndCoh}(\mathcal{X})^{\geq 0}$$

Since  $\mathbf{oblv}^r$  preserves colimits, this t-structure is compatible with filtered colimits. We can describe this t-structure more explicitly using the presentation (4.1), in a way analogous to [GL:IndSch, Lemma 2.4.10] for the t-structure on  $\mathrm{IndCoh}(\mathcal{X})$ .

**Lemma 4.3.2.** *Under the above circumstances, we have:*

- (a) *An object  $\mathcal{F} \in \mathrm{Crys}^r(\mathcal{X})$  belongs to  $\mathrm{Crys}^r(\mathcal{X})^{\geq 0}$  if and only if for every  $\alpha$ , the object  $i_\alpha^!(\mathcal{F}) \in \mathrm{Crys}^r(X_\alpha)$  belongs to  $\mathrm{Crys}^r(X_\alpha)^{\geq 0}$ .*
- (b) *The category  $\mathrm{Crys}^r(\mathcal{X})^{\leq 0}$  is generated under colimits by the essential images of the functors  $(i_\alpha)_{\mathrm{dR},*}(\mathrm{Crys}^r(X_\alpha)^{\leq 0})$ .*

*Proof.* Point (a) follows from the definition and [GL:IndSch, Lemma 2.4.10(a)]. Point (b) follows formally from point (a).  $\square$

4.3.3. Suppose that  $i : X \rightarrow \mathcal{X}$  is a closed embedding of a DG scheme into a DG indscheme. By the exact same argument as in [GL:IndSch, Lemma 2.4.12], we have:

**Lemma 4.3.4.** *The functor  $i_{\mathrm{dR},*}$  is t-exact.*

#### 4.4. Further properties of the left t-structure.

4.4.1. First, let us describe the relationship between the left and the right t-structures on crystals in the case of a smooth classical scheme.

**Proposition 4.4.2.** *Let  $X$  be a smooth classical scheme of dimension  $n$ . Then*

$$\mathcal{F} \in \mathrm{Crys}^l(X)^{\leq 0} \Leftrightarrow \mathcal{F} \in \mathrm{Crys}^r(X)^{\leq -n}.$$

*I.e., the left t-structure agrees with the right t-structure up to a shift by the dimension of  $X$ .*

*Proof.* Recall that the two forgetful functors are related by the commutative diagram

$$\begin{array}{ccc} & \mathrm{Crys}(X) & \\ \mathbf{oblv}_X^l \swarrow & & \searrow \mathbf{oblv}_X^r \\ \mathrm{QCoh}(X) & \xrightarrow{\Psi_X^\vee} & \mathrm{IndCoh}(X) \end{array}$$

By [GL:IndCoh, Proposition 5.7.2], in the case that  $X$  is a smooth classical scheme of dimension  $n$ ,  $\Psi^\vee$  is an equivalence and maps  $\mathrm{QCoh}(X)^{\leq 0}$  isomorphically to  $\mathrm{IndCoh}(X)^{\leq -n}$ . The assertion now follows from Proposition 4.1.3 and Proposition 4.2.5(b).  $\square$

4.4.3. The next proposition compares the “left” and “right” t-structures on  $\mathrm{Crys}(X)$  for an arbitrary DG scheme  $X$ .

**Proposition 4.4.4.** *Let  $X$  be quasi-compact. Then the identity functor on  $\mathrm{Crys}(X)$  has a bounded amplitude when considered in the left or right t-structures.*

*Proof.* Without loss of generality, we can assume that  $X$  is affine. Let  $Z$  be a smooth classical scheme;  $i : X \hookrightarrow Z$  a closed embedding. Let  $U \xrightarrow{j} Z$  denote the complementary open embedding.

Consider the subcategory  $\mathrm{QCoh}(Z)_X \subset \mathrm{QCoh}(Z)$  which is by definition equal to

$$\ker(j^* : \mathrm{QCoh}(Z) \rightarrow \mathrm{QCoh}(U)).$$

Denote by  $\mathbf{e}$  its tautological embedding into  $\mathrm{QCoh}(X)$ , and let  $\mathbf{r}$  denote the right adjoint of  $\mathbf{e}$ , which is given by

$$\mathcal{F} \mapsto \mathrm{Cone}(\mathcal{F} \rightarrow j_* \circ j^*(\mathcal{F}))[-1].$$

Recall the two t-structures that arise on  $\mathrm{QCoh}(Z)_X$ , see [GL:IndSch], Sect. 7.3. The first t-structure is such that the functor  $\mathbf{e}$  is exact. The second t-structure is such that the category  $\mathrm{QCoh}(Z)_X^{\leq 0}$  is generated by the essential image of  $\mathrm{QCoh}(Z)^{\leq 0}$  under  $\mathbf{r}$ .

Considering the formal completion  $Y$  of  $X$  inside  $Z$ , and using the fact that the restriction functor gives an equivalence

$$\mathrm{QCoh}(Z)_X \xrightarrow{\sim} \mathrm{QCoh}(Y)$$

(see [GL:IndSch, Proposition 7.1.3]), we obtain from Propositions 4.1.3, 4.2.7 and 4.4.2, that it suffices to show that the discrepancy between the above two t-structures on  $\mathrm{QCoh}(Z)_X$  is finite.

It is clear that if  $\mathcal{F}$  is  $\leq 0$  in the first t-structure, then it is also  $\leq 0$  in the second. The cohomological amplitude in the other direction is bounded by the amplitude of the functor

$$j_* : \mathrm{QCoh}(U) \rightarrow \mathrm{QCoh}(X).$$

□

4.4.5. Let  $X$  be an arbitrary quasi-compact DG scheme. We have:

**Proposition 4.4.6.**

- (a) The functor  $\mathbf{oblv}_X^l : \mathrm{Crys}(X) \rightarrow \mathrm{QCoh}(X)$  has a bounded cohomological amplitude.
- (b) If  $X$  is eventually coconnective, the functor  $\mathbf{ind}_X^l : \mathrm{QCoh}(X) \rightarrow \mathrm{Crys}(X)$  has a cohomological amplitude bounded from above.

*Remark 4.4.7.* The assumption that  $X$  be eventually coconnective in point (b) is essential. Also, it is easy to show that  $\mathbf{ind}_X^l$  has a cohomological amplitude bounded from below if and only if  $X$  is Gorenstein (see Lemma 4.6.8).

*Proof.* For point (a) we can assume that  $X$  is affine and find a closed embedding  $i : X \hookrightarrow Z$ , where  $Z$  is a smooth classical scheme. In this case, the assertion follows from Proposition 4.2.7 and the fact that the functor

$$i^* : \mathrm{QCoh}(Z) \rightarrow \mathrm{QCoh}(X)$$

has a bounded cohomological amplitude.

Point (b) follows from point (a) by the  $(\mathbf{ind}_X^l, \mathbf{oblv}_X^l)$ -adjunction.

□

**4.5. Left completeness and relation to the naive derived category.**

4.5.1. Let  $X$  be an affine smooth classical scheme. We observe that in this case the category  $\mathrm{Crys}^r(X)$  contains a canonical object

$$\mathfrak{D}_X := \mathbf{ind}_X^r(\mathcal{O}_X),$$

which lies in the heart of the t-structure, and is *projective*, i.e.,

$$H^0(\mathcal{N}) = 0 \Rightarrow \mathrm{Hom}_{\mathrm{Crys}^r(X)}(\mathfrak{D}_X, \mathcal{N}) = 0.$$

Moreover,  $\mathfrak{D}_X$  is a compact generator of  $\mathrm{Crys}^r(X)$ . This implies:

**Corollary 4.5.2.** *Let  $X$  be an affine smooth classical scheme. Then the canonical map*

$$D(\mathrm{Crys}^r(X)^\heartsuit) \rightarrow \mathrm{Crys}^r(X)$$

*is an equivalence. Moreover, the category  $\mathrm{Crys}^r(X)$  is left-complete in its t-structure.*

4.5.3. The above corollary implies left-completeness for any DG scheme  $X$ :

**Corollary 4.5.4.** *For any DG scheme  $X$ , the category  $\mathrm{Crys}^r(X)$  is left-complete in the “right” t-structure.*

*Proof.* The property of being left-complete is Zariski local. Hence, we can assume without loss of generality that  $X$  is affine. Choose a closed embedding  $i : X \hookrightarrow Z$ , where  $Z$  is a smooth classical scheme. Now the assertion follows formally from the fact that the functor  $i_{\mathrm{dR},*}$  is continuous, fully faithful (by Proposition 2.5.6), t-exact (by Proposition 4.2.7), and the fact that  $\mathrm{Crys}^r(Z)$  is left-complete (by the previous corollary).

Here is an alternative argument:

By Corollary 4.2.8, we can assume that  $X$  is eventually coconnective. In this case, the functor  $\mathbf{oblv}_X^l$  commutes with limits, as it admits a left adjoint. Moreover, by Lemma 2.2.6,  $\mathbf{oblv}_X^l$  is conservative, and by Proposition 4.4.6 it has a bounded cohomological amplitude. Therefore, the fact that  $\mathrm{QCoh}(X)$  is left-complete in its t-structure implies the corresponding fact for  $\mathrm{Crys}(X)$ . □

*Remark 4.5.5.* The question of right completeness is not an issue: since our t-structures are compatible with filtered colimits, right completeness is equivalent to the t-structure being separated on the coconnective subcategory, which is evident since  $\mathbf{oblv}_X^r$  is left t-exact and conservative, and the t-structure on  $\mathrm{IndCoh}(X)^+ \xrightarrow{\Psi_X} \mathrm{QCoh}(X)^+$  has this property.

Combining Corollary 4.5.4 with Proposition 4.4.4, we obtain:

**Corollary 4.5.6.** *For a quasi-compact DG scheme  $X$ , the category  $\mathrm{Crys}(X)$  is also left-complete in the “left” t-structure.*

4.5.7. We can also relate the category  $\mathrm{Crys}^r(X)$  to a more classical object.

**Corollary 4.5.8.** *Let  $X$  be an affine DG scheme. Then the canonical map*

$$D(\mathrm{Crys}^r(X)^\heartsuit) \rightarrow \mathrm{Crys}^r(X)$$

*is an equivalence.*

*Proof.* Let us first show that the functor

$$D(\mathrm{Crys}^r(X)^\heartsuit)^+ \rightarrow \mathrm{Crys}^r(X)^+$$

is an equivalence. For that, it suffices to show that every object  $\mathcal{M} \in \mathrm{Crys}^r(X)^\heartsuit$  can be embedded an *injective* object, i.e., an object  $\mathcal{J} \in \mathrm{Crys}^r(X)^\heartsuit$  such that

$$H^0(\mathcal{N}) = 0 \Rightarrow \mathrm{Hom}_{\mathrm{Crys}^r(X)}(\mathcal{N}, \mathcal{J}) = 0.$$

Let  $i : X \hookrightarrow Z$  be a closed embedding, where  $Z$  is a smooth classical scheme. Choose an appropriate embedding  $i_{\mathrm{dR},*}(\mathcal{M}) \hookrightarrow \mathcal{J}$ . The latter exists by Corollary 4.5.2.

Since the functor  $\mathbf{ind}_Z^r$  is exact, we obtain that  $\mathbf{oblv}_Z^r(\mathcal{J})$  is an injective object of  $\mathrm{QCoh}(Z)^\heartsuit$ . This implies that  $\mathcal{J} := i^!(\mathcal{J})$  belongs to  $\mathrm{Crys}^r(X)^\heartsuit$  and has the required property.

It remains to show that  $D(\mathrm{Crys}^r(X)^\heartsuit)$  is left-complete in its t-structure. For that it suffices to exhibit a generator  $\mathcal{P}$  of  $\mathrm{Crys}^r(X)^\heartsuit$  of *bounded Ext dimension*.

Without loss of generality, we can assume that  $X$  is classical. Let

$$\mathcal{P} = \mathfrak{D}_X := \mathbf{ind}_X^r(\mathcal{O}_X).$$

The required property of  $\mathcal{P}$  follows from Proposition 4.2.5(c). □

*Remark 4.5.9.* A standard argument allows us to extend the statement of Corollary 4.5.8 to the case when  $X$  is a quasi-compact DG scheme with an affine diagonal.

*Remark 4.5.10.* Once we identify crystals with D-modules on smooth affine classical schemes, we will obtain many other properties of  $\mathrm{Crys}^r(X)$  on quasi-compact DG schemes: e.g., the fact that the abelian category  $\mathrm{Crys}^r(X)$  is Noetherian and has a finite cohomological dimension with respect to the (either “left” or “right”) t-structure. Note that by Proposition 4.2.7, in order to establish both these properties, it suffices to show them for affine smooth classical schemes.

#### 4.6. The “coarse” induction and forgetful functors.

4.6.1. Let  $X$  be a DG scheme. Recall that the functor  $\Psi_X$  identifies the category  $\mathrm{QCoh}(X)$  with the left-completion of  $\mathrm{IndCoh}(X)$  (this follows from [GL:IndCoh, Proposition 1.2.2] and the fact that  $\mathrm{QCoh}(X)$  is left-complete in its t-structure).

Since the category  $\mathrm{Crys}^r(X)$  is left-complete, and the functor  $\mathbf{ind}_X^r$  is exact, by the universal property of left completions, we obtain:

**Corollary 4.6.2.** *The functor  $\mathbf{ind}_X^r$  canonically factors as*

$$\mathrm{IndCoh}(X) \xrightarrow{\Psi_X} \mathrm{QCoh}(X) \xrightarrow{\mathbf{ind}_X^r} \mathrm{Crys}^r(X).$$

We can also consider the functor

$$\mathbf{oblv}_X^r : \mathrm{Crys}^r(X) \rightarrow \mathrm{QCoh}(X),$$

given by  $\Psi_X \circ \mathbf{oblv}_X^r$ .

**Proposition 4.6.3.** *When  $X$  is eventually coconnective, the functors  $(\mathbf{ind}_X^r, \mathbf{oblv}_X^r)$  form an adjoint pair.*

*Proof.* This follows formally from the fact that for  $X$  eventually coconnective, the functor  $\Psi_X$  admits a fully faithful left adjoint  $\Xi_X$ . In particular,

$$\mathbf{ind}_X^r \simeq \mathbf{ind}_X^r \circ \Xi_X.$$

□

*Remark 4.6.4.* The functors  $(\mathbf{ind}_X^r, \mathbf{oblv}_X^r)$  are *not* adjoint unless  $X$  is eventually coconnective. (Otherwise, the functor  $\mathbf{ind}_X^r$  does not preserve compact objects; for instance, it sends  $\mathcal{O}_X$  to a non-compact object of  $\mathrm{Crys}^r(X)$ .)

**Proposition 4.6.5.** *The functor  $\mathbf{oblv}_X^r$  is conservative.*

*Proof.* This follows from the fact that  $\mathrm{QCoh}(X)$  is left-complete and  $\mathbf{oblv}_X^r$  has a finite cohomological amplitude.  $\square$

4.6.6. *Warning.* Let  $X$  be an eventually coconnective DG scheme, and consider the pair of adjoint functors

$$\Xi_X : \mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}(X) : \Psi_X$$

with  $\Xi_X$  being fully faithful (see [GL:IndCoh, Sect. 1.4]).

We have seen that the functor  $\mathbf{ind}_X^r$  factors through the colocalization functor  $\Psi_X$ . However, it is *not* true in general that the functor  $\mathbf{oblv}_X^r$  factors through  $\Xi_X$ , i.e., that it takes values in  $\mathrm{QCoh}(X)$ , considered as a full subcategory of  $\mathrm{IndCoh}(X)$  via  $\Xi_X$ .

4.6.7. In fact, the following holds:

**Lemma 4.6.8** (Drinfeld). *The functor  $\mathbf{oblv}_X^r$  factors through the essential image of  $\mathrm{QCoh}(X)$  under  $\Xi_X$  if and only if  $X$  is Gorenstein.*

Recall that a DG scheme  $X$  is said to be Gorenstein if:

- (a)  $\omega_X \in \mathrm{Coh}(X)$  (which is equivalent to  $X$  being eventually coconnective, see [GL:IndCoh, Proposition 8.5.6]);
- (b) When considered as a coherent sheaf,  $\omega_X$  is a graded line bundle (which is equivalent to  $\omega_X \in \mathrm{QCoh}(X)^{\mathrm{perf}}$ , see [GL:IndCoh, Proposition 8.5.9]).

*Proof.* One implication is immediate: suppose that  $\mathbf{oblv}_X^r$  factors through  $\mathrm{QCoh}(X)$ . In particular, we obtain that  $\omega_X \in \mathrm{IndCoh}(X)$  lies in the essential image of  $\Xi_X$ . But we have the following general observation: if  $\mathcal{F} \in \mathrm{Coh}(X) \subset \mathrm{IndCoh}(X)$  lies in the essential image of  $\Xi_X$ , then  $\mathcal{F} \in \mathrm{QCoh}(X)^{\mathrm{perf}}$ . Indeed, since  $\Xi_X$  is fully faithful and continuous, if an object  $\mathcal{F}' \in \mathrm{QCoh}(X)$  is such that  $\mathcal{F} = \Xi_X(\mathcal{F}')$  is compact, then  $\mathcal{F}'$  is compact.

For the opposite implication, we write  $\mathbf{oblv}_X^r(\mathcal{M})$  as

$$\Psi_X^\vee(\mathcal{M}) = \mathbf{oblv}_X^l(\mathcal{M}) \otimes \omega_X,$$

where the tensor product is understood in the sense of the action of  $\mathrm{QCoh}(X)$  on  $\mathrm{IndCoh}(X)$ , see [GL:IndCoh], Sect. 1.3. Recall also that the functor  $\Xi_X$  is tautologically compatible with the above action of  $\mathrm{QCoh}(X)$ . Hence, if  $\omega_X$ , being perfect, lies in the essential image of  $\Xi_X$ , then so does  $\mathbf{oblv}_X^l(\mathcal{M}) \otimes \omega_X$ .  $\square$

**4.7. Crystals and D-modules.** Let  $X$  be a classical scheme of finite type. We will show that the category  $\mathrm{Crys}^r(X)$  is canonically identified with the category  $\mathrm{D-mod}^r(X)$  of right D-modules on  $X$ .

*Remark 4.7.1.* The category  $\mathrm{D-mod}^r(X)$  satisfies Zariski descent. Therefore, in what follows, by Proposition 3.1.2, it will suffice to establish a canonical equivalence for affine schemes.

4.7.2. Let  $Z$  be a smooth classical affine scheme, and let  $i : X \hookrightarrow Z$  be a closed embedding. By the classical Kashiwara's lemma and Proposition 2.5.6, in order to construct an equivalence

$$\mathrm{Crys}^r(X) \simeq \mathrm{D}\text{-mod}^r(X),$$

it suffices to do so for  $Z$ .

Hence, we can assume that  $X$  itself is a smooth classical affine scheme. We shall construct the equivalence in question together with the commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Crys}^r(X) & \longrightarrow & \mathrm{D}\text{-mod}^r(X) \\ \mathrm{oblv}_X^r \downarrow & & \downarrow \\ \mathrm{IndCoh}(X) & \xrightarrow{\Psi_X} & \mathrm{QCoh}(X), \end{array}$$

where the right vertical arrow is the natural forgetful functor, and the functor  $\Psi_X$  is the equivalence of [GL:IndCoh, Lemma 1.1.6].

By Proposition 2.4.4, this is the same as constructing an equivalence between *left* crystals and *left* D-modules together with the commutative diagram of functors

$$(4.2) \quad \begin{array}{ccc} \mathrm{Crys}^l(X) & \xrightarrow{\quad} & \mathrm{D}\text{-mod}^l(X) \\ & \searrow \mathrm{oblv}_X^l & \swarrow \\ & \mathrm{QCoh}(X) & \end{array}$$

4.7.3. By Corollary 4.5.2, Propositions 2.4.4 and 4.4.2, the category  $\mathrm{Crys}^l(X)$  identifies with the derived category of the heart of its t-structure. The category  $\mathrm{D}\text{-mod}^l(X)$  is by definition the derived category of  $\mathrm{D}\text{-mod}^l(X)^\heartsuit$ . Moreover, the vertical arrows in Diagram (4.2) are t-exact.

Hence, it suffices to construct the desired equivalence at the level of the corresponding abelian categories

$$(4.3) \quad \begin{array}{ccc} \mathrm{Crys}^l(X)^\heartsuit & \longrightarrow & \mathrm{D}\text{-mod}^l(X)^\heartsuit \\ \mathrm{oblv}_X^l \downarrow & & \downarrow \\ \mathrm{QCoh}(X)^\heartsuit & \xrightarrow{\mathrm{Id}} & \mathrm{QCoh}(X)^\heartsuit \end{array}$$

The latter is a classical calculation, due to Grothendieck. Below we reproduce it for the sake of completeness.

4.7.4. The abelian categories  $\mathrm{Crys}^l(X)^\heartsuit$  and  $\mathrm{D}\text{-mod}^l(X)^\heartsuit$  are given as modules over the monads  $M_{\mathrm{Crys}(X)}$  and  $M_{\mathrm{D}\text{-mod}(X)}$  in the category  $\mathrm{QCoh}(X)^\heartsuit$ . We have that  $M_{\mathrm{D}\text{-mod}(X)}$  is given by the algebra object  $\mathfrak{D}_X^l$ .

4.7.5. The category  $\mathrm{Crys}^l(X)^\heartsuit$  is given by the heart of the category of quasi-coherent sheaves on the truncated simplicial object

$$(X \times X \times X)_X^\wedge \begin{array}{c} \xrightarrow{p_{12}} \\ \xrightarrow{p_{13}} \\ \xrightarrow{p_{23}} \end{array} (X \times X)_X^\wedge \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} X$$

Explicitly, an object of  $\mathrm{Crys}^l(X)^\heartsuit$  is a quasi-coherent sheaf  $\mathcal{F} \in \mathrm{QCoh}(X)^\heartsuit$  together with an isomorphism

$$\phi : p_2^*(\mathcal{F}) \xrightarrow{\sim} p_1^*(\mathcal{F})$$

which restricts to the identity on the diagonal and satisfies the cocycle condition

$$(4.4) \quad p_{13}^*(\phi) = p_{12}^*(\phi) \circ p_{23}^*(\phi).$$

By adjunction, the isomorphism  $p_2^*(\mathcal{F}) \xrightarrow{\sim} p_1^*(\mathcal{F})$  gives a map

$$\check{a} : \mathcal{F} \rightarrow p_{2*}(p_1^*(\mathcal{F})) \simeq p_{2*}(p_1^*(\mathcal{O}_X)) \otimes_{\mathcal{O}_X} \mathcal{F},$$

where we regard  $p_{2*}(p_1^*(\mathcal{O}_X))$  as a pro- $\mathcal{O}_X$ -bimodule. But by Grothendieck's definition of the sheaf of differential operators,  $p_{2*}(p_1^*(\mathcal{O}_X))$  is the  $\mathcal{O}_X$ -linear dual of  $\text{Diff}_X$ . This gives a map

$$a : \text{Diff}_X \otimes \mathcal{F} \rightarrow \mathcal{F}.$$

Now, we claim that the cocycle condition is equivalent to the commutativity of the following diagram

$$(4.5) \quad \begin{array}{ccc} \text{Diff}_X \otimes \text{Diff}_X \otimes \mathcal{F} & \xrightarrow{m \otimes \text{id}_{\mathcal{F}}} & \text{Diff}_X \otimes \mathcal{F} \\ \text{id}_{\text{Diff}_X} \otimes a \downarrow & & \downarrow a \\ \text{Diff}_X \otimes \mathcal{F} & \xrightarrow{a} & \mathcal{F} \end{array}$$

where  $m : \text{Diff}_X \otimes \text{Diff}_X \rightarrow \text{Diff}_X$  is multiplication of differential operators. In other words, the cocycle condition implies that the map  $a$  makes  $\mathcal{F}$  into a left  $\text{Diff}_X$ -module. Recall that multiplication of differential operators is dual to the map

$$\check{m} : p_{2*}(p_1^*(\mathcal{O}_X)) \rightarrow p_{2*} \circ p_{13*} \circ p_{12}^* \circ p_1^*(\mathcal{O}_X) \simeq p_{2*} \circ p_{13*} \circ p_{13}^* \circ p_1^*(\mathcal{O}_X)$$

given by the unit of the  $(p_{13}^*, p_{13*})$  adjunction. Thus, the commutativity of (4.5) is equivalent to the commutativity of

$$(4.6) \quad \begin{array}{ccc} \mathcal{F} & \xrightarrow{\check{a}} & p_{2*}(p_1^*(\mathcal{F})) \\ \check{a} \downarrow & & \downarrow \text{id}_{p_{2*}(p_1^*(\mathcal{O}_X))} \otimes \check{a} \\ p_{2*}(p_1^*(\mathcal{F})) & \xrightarrow{\check{m} \otimes \text{id}_{\mathcal{F}}} & p_{2*} \circ p_{13*} \circ p_{13}^* \circ p_1^*(\mathcal{F}) \end{array}$$

Note that the composite

$$\mathcal{F} \xrightarrow{\check{a}} p_{2*}(p_1^*(\mathcal{F})) \xrightarrow{\text{id}_{p_{2*}(p_1^*(\mathcal{O}_X))} \otimes \check{a}} p_{2*} \circ p_{13*} \circ p_{13}^* \circ p_1^*(\mathcal{F})$$

is the adjoint of the isomorphism

$$p_{12}^*(\phi) \circ p_{23}^*(\phi) : p_{13}^*(p_2^*(\mathcal{F})) \xrightarrow{\sim} p_{13}^*(p_1^*(\mathcal{F})).$$

Furthermore, we have that the composite

$$\mathcal{F} \xrightarrow{\check{a}} p_{2*}(p_1^*(\mathcal{F})) \xrightarrow{\check{m} \otimes \text{id}_{\mathcal{F}}} p_{2*} \circ p_{13*} \circ p_{13}^* \circ p_1^*(\mathcal{F})$$

is the adjoint of

$$p_{13}^*(\phi) : p_{13}^*(p_2^*(\mathcal{F})) \xrightarrow{\sim} p_{13}^*(p_1^*(\mathcal{F})).$$

This follows from the factorization

$$\begin{array}{ccc} p_2^*(\mathcal{F}) & \xrightarrow{\quad} & p_{13*} \circ p_{13}^* \circ p_1^*(\mathcal{F}) \\ & \searrow \phi & \nearrow \text{unit} \\ & p_1^*(\mathcal{F}) & \end{array}$$

of the adjoint of  $p_{13}^*(\phi)$ . Thus by adjunction, the cocycle condition (4.4) is equivalent to the commutativity of (4.6). Hence, we have that  $\mathcal{F}$  has a natural structure of a left  $\mathrm{Diff}_X$ -module.

Thus, we have constructed a functor

$$\mathrm{Crys}^l(X)^\heartsuit \rightarrow \mathrm{D-mod}^l(X)^\heartsuit$$

which is compatible with the forgetful functors to  $\mathrm{QCoh}(X)^\heartsuit$ . This gives a homomorphism of monads

$$M_{\mathrm{Crys}(X)} \rightarrow M_{\mathrm{D-mod}(X)}.$$

It suffices to show that this homomorphism is an isomorphism of functors. However, this follows immediately from Lemma 3.4.7.

4.7.6. Note that by construction, the equivalence

$$\mathrm{Crys}^l(X) \rightarrow \mathrm{D-mod}^l(X)$$

is compatible with pull-back for maps  $f : Y \rightarrow X$  between smooth classical schemes.

## 5. TWISTINGS

In this section, we do *not* assume that the prestacks and DG schemes that we consider are locally almost of finite type. We will resume this assumption in Sect. 5.7.

### 5.1. Gerbes.

5.1.1. Let  $B\mathbb{G}_m$  be the classifying stack of the group  $\mathbb{G}_m$ . In other words,  $B\mathbb{G}_m$  is the algebraic stack which represents the functor which assigns to an affine DG scheme  $S$ , the  $\infty$ -groupoid of line bundles on  $S$ . In fact, since  $\mathbb{G}_m$  is an abelian group, the stack  $B\mathbb{G}_m$  has a natural abelian group structure. The multiplication map on  $B\mathbb{G}_m$  represents tensor product of line bundles. This structure upgrades  $B\mathbb{G}_m$  to a functor from affine DG schemes to  $\infty$ -Picard groupoids, i.e. connective spectra.

For our purposes, a  $\mathbb{G}_m$ -gerbe will be a  $B\mathbb{G}_m$ -torsor  $G$ , which satisfies any of the following three (non-equivalent) conditions:

- (i)  $G$  is locally non-empty in the étale topology <sup>5</sup>.
- (ii)  $G$  is locally non-empty in the Zariski topology.
- (iii)  $G$  is globally non-empty.

Specifically, let  $B^{\mathrm{naive}}(B\mathbb{G}_m)$  be the classifying prestack of  $B\mathbb{G}_m$ . It is given by the geometric realization of the simplicial prestack

$$B^{\mathrm{naive}}(B\mathbb{G}_m) := \left| \cdots B\mathbb{G}_m \times B\mathbb{G}_m \rightrightarrows B\mathbb{G}_m \rightrightarrows \mathrm{pt} \right|.$$

Let  $B^{\mathrm{Zar}}(B\mathbb{G}_m)$  (resp.  $B^2\mathbb{G}_m$ ) be the Zariski (resp. étale) sheafification of  $B^{\mathrm{naive}}B\mathbb{G}_m$ . The prestacks  $B^2\mathbb{G}_m$ ,  $B^{\mathrm{Zar}}(B\mathbb{G}_m)$  and  $B^{\mathrm{naive}}(B\mathbb{G}_m)$  represent  $\mathbb{G}_m$ -gerbes satisfying the above conditions (i), (ii), and (iii) respectively.

Let  $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{DGSch}^{\mathrm{aff}}}$  be the functor

$$(\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-PicGrpd}$$

which associates to an affine DG scheme  $S$ , the groupoid of  $\mathbb{G}_m$ -gerbes, where we consider either of the three notions of gerbe defined above. While these three versions do not give equivalent

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<sup>5</sup>By Toën's theorem, this is equivalent to local non-emptiness in the fppf topology.



notions of  $\mathbb{G}_m$ -gerbes, we will see shortly that they do lead to the same definition of twisting, since the relevant gerbes will be those whose restrictions to  ${}^{cl,red}S$  are trivialized.

5.1.2. We define the functor

$$(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{PreStk}} : (\mathrm{PreStk})^{op} \rightarrow \infty\text{-PicGrpd}$$

as the right Kan extension of  $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{DGSch}^{\mathrm{aff}}}$  along

$$(\mathrm{DGSch}^{\mathrm{aff}})^{op} \hookrightarrow \mathrm{PreStk}.$$

I.e., for  $\mathcal{Y} \in \mathrm{PreStk}$ ,

$$\mathrm{Ge}_{\mathbb{G}_m}(\mathcal{Y}) := \lim_{S \in (\mathrm{DGSch}_{/\mathcal{Y}}^{\mathrm{aff}})^{op}} \mathrm{Ge}_{\mathbb{G}_m}(S).$$

Thus, informally, a  $\mathbb{G}_m$ -gerbe on  $\mathcal{Y}$  is an assignment of a  $\mathbb{G}_m$ -gerbe on every  $S \in \mathrm{DGSch}^{\mathrm{aff}}$  mapping to  $\mathcal{Y}$ , functorial in  $S$ .

For a subcategory  $\mathbf{C} \subset \mathrm{PreStk}$ , let  $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathbf{C}}$  denote the restriction  $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{PreStk}}|_{\mathbf{C}}$ .

5.1.3. Recall the notion of convergence for a functor

$$\Phi : (\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \mathbf{D},$$

where  $\mathbf{D}$  is a category with limits (see [GL:Stacks], Sect. 1.2.1). By definition, such a functor is convergent if the map

$$\Phi(S) \rightarrow \lim_n \Phi(\tau^{\leq n}(S))$$

is an isomorphism, where  $\tau^{\leq n}(S)$  denotes the  $n$ -coconnective truncation of  $S$ .

The convergence of  $\mathbb{G}_m$  as a prestack implies:

**Lemma 5.1.4.** *The functor  $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{DGSch}^{\mathrm{aff}}}$  is convergent.*

We can tautologically reformulate this lemma as follows:

**Corollary 5.1.5.** *The functor  $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{PreStk}}$  maps isomorphically to the right Kan extension of  $(\mathrm{Ge}_{\mathbb{G}_m})_{<\infty\mathrm{DGSch}^{\mathrm{aff}}}$  along*

$$(<\infty\mathrm{DGSch}^{\mathrm{aff}})^{op} \hookrightarrow (\mathrm{DGSch}^{\mathrm{aff}})^{op} \hookrightarrow (\mathrm{PreStk})^{op}.$$

5.1.6. *Gerbes in the locally almost of finite type case.* The next lemma says that for  $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ , we “do not need to know” about DG schemes that are not locally almost of finite type in order to know what gerbes on  $\mathcal{Y}$  are:

**Lemma 5.1.7.** *For  $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ , the naturally defined map*

$$\mathrm{Ge}_{\mathbb{G}_m}(\mathcal{Y}) \rightarrow \lim_{S \in ((<\infty\mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{Y}})^{op}} \mathrm{Ge}_{\mathbb{G}_m}(S)$$

*is an equivalence.*

*Proof.* The assertion is equivalent to the statement that the prestacks

$$B^{\mathrm{naive}}(B\mathbb{G}_m), B^{\mathrm{Zar}}(B\mathbb{G}_m) \text{ and } B^2(\mathbb{G}_m)$$

are almost locally of finite type. Since  $B\mathbb{G}_m$  is an object of  $\mathrm{PreStk}_{\mathrm{lft}}$ ,  $B^{\mathrm{naive}}(B\mathbb{G}_m)$  is locally almost of finite type. Furthermore, by [GL:Stacks, Sect. 2.5], the prestacks  $B^{\mathrm{Zar}}(B\mathbb{G}_m)$  and  $B^2\mathbb{G}_m$  are also locally almost of finite type.  $\square$

As a corollary, we obtain:

**Corollary 5.1.8.** *The functor  $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{PreStk}_{\mathrm{laft}}}$  maps isomorphically to the right Kan extension of  $(\mathrm{Ge}_{\mathbb{G}_m})_{<\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}}$  along the inclusions*

$$(<\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}})^{op} \hookrightarrow (\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}})^{op} \hookrightarrow \mathrm{PreStk}_{\mathrm{laft}}^{op}.$$

## 5.2. The notion of twisting.

5.2.1. Let  $\mathcal{Y}$  be a prestack. The Picard groupoid of twistings on  $\mathcal{Y}$  defined as

$$\mathrm{Tw}(\mathcal{Y}) := \ker(p_{\mathrm{dR}, \mathcal{Y}}^* : \mathrm{Ge}_{\mathbb{G}_m}(\mathcal{Y}_{\mathrm{dR}}) \rightarrow \mathrm{Ge}_{\mathbb{G}_m}(\mathcal{Y})).$$

5.2.2. Informally, a twisting  $T$  on  $\mathcal{Y}$  is the following data: for every  $S \in \mathrm{DGSch}^{\mathrm{aff}}$  equipped with a map  $^{cl, red} S \rightarrow \mathcal{Y}$  we specify an object  $G_S \in \mathrm{Ge}_{\mathbb{G}_m}(S)$ , which behaves compatibly under the maps  $S_1 \rightarrow S_2$ . Additionally, for every extension of the above map to a map  $S \rightarrow \mathcal{Y}$  we specify a trivialization of  $G_S$ , which also behaves functorially with respect to maps  $S_1 \rightarrow S_2$ .

5.2.3. *Example.* Let  $\mathcal{L}$  be a line bundle on  $\mathcal{Y}$ . We define a twisting  $T(\mathcal{L})$  on  $\mathcal{Y}$  as follows: it assigns to every  $S \in \mathrm{DGSch}^{\mathrm{aff}}$  with a map  $^{cl, red} S \rightarrow \mathcal{Y}$  the trivial  $\mathbb{G}_m$ -gerbe. For a map  $S \rightarrow \mathcal{Y}$ , we trivialize the above gerbe by multiplying the tautological trivialization by  $\mathcal{L}$ .

5.2.4. It is clear that twistings form a functor

$$\mathrm{Tw}_{\mathrm{PreStk}} : \mathrm{PreStk}^{op} \rightarrow \infty\text{-PicGrpd}.$$

For a morphism  $f : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  we let  $f^*$  denote the corresponding functor

$$\mathrm{Tw}(\mathcal{Y}_2) \rightarrow \mathrm{Tw}(\mathcal{Y}_1).$$

If  $\mathbf{C}$  is a subcategory of  $\mathrm{PreStk}$  (e.g.,  $\mathbf{C} = \mathrm{DGSch}^{\mathrm{aff}}$  or  $\mathrm{DGSch}$ ), we let  $\mathrm{Tw}_{\mathbf{C}}$  denote the restriction of  $\mathrm{Tw}_{\mathrm{PreStk}}$  to  $\mathbf{C}^{op}$ .

5.2.5. By construction, the functor  $\mathrm{Tw}_{\mathrm{PreStk}}$  takes colimits in  $\mathrm{PreStk}$  to limits in  $\infty\text{-PicGrpd}$ . Hence, from Corollary 1.1.5, we obtain:

**Lemma 5.2.6.** *The functor  $\mathrm{Tw}_{\mathrm{PreStk}}$  maps isomorphically to the right Kan extension of  $\mathrm{Tw}_{\mathbf{C}}$  along*

$$\mathbf{C}^{op} \hookrightarrow \mathrm{PreStk}^{op}$$

for  $\mathbf{C}$  being one of the categories

$$\mathrm{DGSch}^{\mathrm{aff}}, \mathrm{DGSch}_{\mathrm{qc-qc}}, \mathrm{DGSch}.$$

Concretely, this lemma says that the map

$$\mathrm{Tw}(\mathcal{Y}) \rightarrow \lim_{S \in (\mathrm{DGSch}_{\mathcal{Y}}^{\mathrm{aff}})^{op}} \mathrm{Tw}(S)$$

is an isomorphism (and that  $\mathrm{DGSch}^{\mathrm{aff}}$  can be replaced by  $\mathrm{DGSch}_{\mathrm{qc-qc}}$  or  $\mathrm{DGSch}$ .)

Informally, this means that to specify a twisting on a prestack  $\mathcal{Y}$  is equivalent to specifying a compatible family of twistings on affine DG schemes  $S$  mapping to  $\mathcal{Y}$ .

5.2.7. Note also that from Lemma 5.1.4, we obtain:

**Corollary 5.2.8.**

(a) *The functor*

$$\mathrm{Tw}_{\mathrm{DGSch}^{\mathrm{aff}}} : (\mathrm{DGSch}^{\mathrm{aff}})^{\mathrm{op}} \rightarrow \infty\text{-PicGrpd}$$

*is convergent.*

(b) *The functor  $\mathrm{Tw}_{\mathrm{PreStk}}$  maps isomorphically to the right Kan extension of*

$$\mathrm{Tw}_{<\infty\mathrm{DGSch}^{\mathrm{aff}}} := \mathrm{Tw}_{\mathrm{DGSch}^{\mathrm{aff}}} \big|_{<\infty\mathrm{DGSch}^{\mathrm{aff}}}$$

*along*

$$<\infty\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{PreStk}.$$

5.2.9. *Twistings in the locally almost of finite type case.* As in the case of gerbes, we “do not need to know” about DG schemes that are not locally almost of finite type in order to know what twistings on  $\mathcal{Y}$  if  $\mathcal{Y}$  is locally almost of finite type.

**Corollary 5.2.10.**

(a) *For  $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}$ , the naturally defined map*

$$\mathrm{Tw}(\mathcal{Y}) \rightarrow \lim_{S \in ((<\infty\mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{Tw}(S)$$

*is an equivalence.*

(b) *The functor  $\mathrm{Tw}_{\mathrm{PreStk}_{\mathrm{lft}}}$  maps isomorphically to the right Kan extension of  $\mathrm{Tw}_{<\infty\mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}}$  along the inclusions*

$$(<\infty\mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow (\mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}})^{\mathrm{op}} \hookrightarrow \mathrm{PreStk}_{\mathrm{lft}}^{\mathrm{op}}.$$

**5.3. Twisting and the infinitesimal groupoid.**

5.3.1. Let

$$\mathcal{Y}^1 \rightrightarrows \mathcal{Y}^0$$

be a groupoid object in  $\mathrm{PreStk}$ , and let  $\mathcal{Y}^\bullet$  be the corresponding simplicial object. Let us recall the notion of central extension of this groupoid object by  $\mathbb{G}_m$ . (Here  $\mathbb{G}_m$  can be replaced by any commutative group-object  $H \in \mathrm{PreStk}$ ).

By definition, a central extension of  $\mathcal{Y}^1 \rightrightarrows \mathcal{Y}^0$  by  $\mathbb{G}_m$  is a map of simplicial prestacks

$$\mathcal{Y}^\bullet \rightarrow \mathrm{pt}^\bullet / B^{\mathrm{naive}}\mathbb{G}_m$$

where  $\mathrm{pt}^\bullet / B^{\mathrm{naive}}\mathbb{G}_m$  is the Čech nerve of the trivial  $\mathbb{G}_m$  gerbe  $\mathrm{pt} \rightarrow B^{\mathrm{naive}}\mathbb{G}_m$ , i.e. the simplicial prestack

$$\cdots B\mathbb{G}_m \times B\mathbb{G}_m \rightrightarrows B\mathbb{G}_m \rightrightarrows \mathrm{pt}$$

Equivalently, such a central extension is an object of  $\mathrm{Ge}_{\mathbb{G}_m}(|\mathcal{Y}^\bullet|)$ , equipped with a trivialization of its restriction under

$$\mathcal{Y}^0 \rightarrow |\mathcal{Y}^\bullet|.$$

Informally, the data of such a central extension is a line bundle  $\mathcal{L}$  on  $\mathcal{Y}^1$ , whose pullback under the degeneracy map  $\mathcal{Y}^0 \rightarrow \mathcal{Y}^1$  is trivialized, and such that for the three maps

$$p_{1,2}, p_{2,3}, p_{1,2} : \mathcal{Y}^2 \rightarrow \mathcal{Y}^1,$$

we are given an isomorphism

$$p_{1,2}^*(\mathcal{L}) \otimes p_{2,3}^*(\mathcal{L}) \simeq p_{1,3}^*(\mathcal{L}),$$

and such that the further coherence conditions are satisfied.

5.3.2. For  $\mathcal{Y} \in \text{PreStk}$  consider its infinitesimal groupoid

$$\mathcal{Y} \times_{\mathcal{Y}_{\text{dR}}} \mathcal{Y} \rightrightarrows \mathcal{Y}.$$

By definition, a twisting on  $\mathcal{Y}$  gives rise to a central extension of its infinitesimal groupoid by  $\mathbb{G}_m$ . Conversely, from Lemma 1.2.4, we obtain:

**Corollary 5.3.3.** *Assume that  $\mathcal{Y}$  is classically formally smooth. Then the above functor*

$$\text{Tw}(\mathcal{Y}) \rightarrow \{\text{Central extensions of the infinitesimal groupoid of } \mathcal{Y} \text{ by } \mathbb{G}_m\}$$

*is an equivalence.*

#### 5.4. Variant: other structure groups.

5.4.1. Let  $S$  be an affine DG scheme. Consider the Picard groupoid

$$\text{Ge}_{\mathbb{G}_m}^{\text{red}}(S) := \ker \left( \text{Ge}_{\mathbb{G}_m}(S) \rightarrow \text{Ge}_{\mathbb{G}_m}^{\text{cl,red}}(S) \right).$$

Let  $(\text{Ge}_{\mathbb{G}_m}^{\text{red}})_{\text{DGSch}^{\text{aff}}}$  denote the resulting functor

$$(\text{DGSch})^{op} \rightarrow \infty\text{-PicGrpd}.$$

5.4.2. By definition, we can think of  $\text{Ge}_{\mathbb{G}_m}^{\text{red}}(S)$  as gerbes (in any of the three versions of Sect. 5.1.1) with respect to the sheaf of abelian groups

$$(\mathcal{O}^\times)_S^{\text{red}} : \ker(\mathcal{O}_S^\times \rightarrow \mathcal{O}_{\text{cl,red } S}^\times).$$

5.4.3. In addition to  $\mathbb{G}_m$ -gerbes, we can also consider  $\mathbb{G}_a$ -gerbes. We have the functor

$$(\text{Ge}_{\mathbb{G}_a})_{\text{DGSch}^{\text{aff}}} : (\text{DGSch}^{\text{aff}})^{op} \rightarrow \infty\text{-PicGrpd}$$

which assigns to an affine DG scheme  $S$  the groupoid of  $\mathbb{G}_a$ -gerbes on  $S$ . Note that unlike the case of  $\mathbb{G}_m$ -gerbes, the three notions of gerbes discussed in Sect. 5.1.1 are equivalent for  $\mathbb{G}_a$ -gerbes. Thus, we have that  $(\text{Ge}_{\mathbb{G}_a})_{\text{DGSch}^{\text{aff}}}$  is represented by  $B^2\mathbb{G}_a$ , which is the geometric realization of the corresponding simplicial prestack.

By definition, for an affine DG scheme  $S$ , we have

$$\text{Ge}_{\mathbb{G}_a}(S) = B^2(\mathcal{O}_S).$$

In particular, viewed as a connective spectrum,  $\text{Ge}_{\mathbb{G}_a}(S)$  has a natural structure of a module over the ground field  $k$ . This upgrades  $(\text{Ge}_{\mathbb{G}_a})_{\text{DGSch}^{\text{aff}}}$  to a functor

$$(\text{DGSch})^{op} \rightarrow \infty\text{-PicGrpd}_k,$$

where  $\infty\text{-PicGrpd}_k$  denotes the category of  $k$ -modules in connective spectra. Note that by the Dold-Kan correspondence, we have

$$\infty\text{-PicGrpd}_k \simeq \text{Vect}^{\leq 0}.$$

We define the functor

$$(\text{Ge}_{\mathbb{G}_a})_{\text{PreStk}} : \text{PreStk}^{op} \rightarrow \infty\text{-PicGrpd}_k$$

as the right Kan extension of the functor  $(\text{Ge}_{\mathbb{G}_a})_{\text{DGSch}^{\text{aff}}}$  along

$$(\text{DGSch}^{\text{aff}})^{op} \hookrightarrow \text{PreStk}^{op}.$$

5.4.4. As with  $\mathbb{G}_m$ -gerbes, we can consider the Picard groupoid

$$\mathrm{Ge}_{\mathbb{G}_a}^{\text{red}}(S) := \ker \left( \mathrm{Ge}_{\mathbb{G}_a}(S) \rightarrow \mathrm{Ge}_{\mathbb{G}_a}^{\text{cl,red}}(S) \right),$$

and let  $(\mathrm{Ge}_{\mathbb{G}_a}^{\text{red}})_{\mathrm{DGSch}^{\mathrm{aff}}}$  denote the resulting functor

$$(\mathrm{DGSch})^{op} \rightarrow \infty\text{-PicGrpd}_k.$$

By definition, for an affine DG scheme  $S$ ,  $\mathrm{Ge}_{\mathbb{G}_a}^{\text{red}}(S)$  is given by gerbes for the sheaf of abelian groups

$$\mathcal{O}_S^{\text{red}} := \ker(\mathcal{O}_S \rightarrow \mathcal{O}_{\mathrm{cl,red}S}).$$

Explicitly, we have

$$\mathrm{Ge}_{\mathbb{G}_a}^{\text{red}}(S) \simeq B^2(\mathcal{O}_S^{\text{red}}).$$

5.4.5. Recall from [GL:IndSch, Sect. 6.7.6] that the exponential map defines an isomorphism

$$\exp : \mathcal{O}_S^{\text{red}} \rightarrow (\mathcal{O}^\times)_S^{\text{red}}.$$

This gives an isomorphism of functors

$$(5.1) \quad \exp : (\mathrm{Ge}_{\mathbb{G}_a}^{\text{red}})_{\mathrm{DGSch}^{\mathrm{aff}}} \rightarrow (\mathrm{Ge}_{\mathbb{G}_m}^{\text{red}})_{\mathrm{DGSch}^{\mathrm{aff}}}.$$

Thus, if we realize  $\mathrm{Ge}_{\mathbb{G}_m}^{\text{red}}(S)$  as gerbes in the étale or Zariski topology, this category has a trivial  $\pi_0$  and  $\pi_1$ . In other words, any gerbe on an affine DG scheme is globally non-empty, and any automorphism is non-canonically isomorphic to identity. Thus, for  $\mathrm{Ge}_{\mathbb{G}_m}^{\text{red}}$ , the three notions of gerbe from Sect. 5.1.1 coincide.

5.4.6. The isomorphism (5.1) endows  $\mathrm{Ge}_{\mathbb{G}_m}^{\text{red}}(S)$ , viewed as a connective spectrum, with a structure of module over the ground field  $k$ . This upgrades  $(\mathrm{Ge}_{\mathbb{G}_m}^{\text{red}})_{\mathrm{DGSch}^{\mathrm{aff}}}$  to a functor

$$(\mathrm{DGSch})^{op} \rightarrow \infty\text{-PicGrpd}_k.$$

We define the functor

$$(\mathrm{Ge}_{\mathbb{G}_m}^{\text{red}})_{\mathrm{PreStk}} : \mathrm{PreStk}^{op} \rightarrow \infty\text{-PicGrpd}_k$$

as the right Kan extension of the functor  $(\mathrm{Ge}_{\mathbb{G}_m}^{\text{red}})_{\mathrm{DGSch}^{\mathrm{aff}}}$  along

$$(\mathrm{DGSch}^{\mathrm{aff}})^{op} \hookrightarrow \mathrm{PreStk}^{op}.$$

5.4.7. By definition, for  $\mathcal{Y} \in \mathrm{PreStk}$

$$\mathrm{Ge}_{\mathbb{G}_m}^{\text{red}}(\mathcal{Y}) := \lim_{S \in (\mathrm{DGSch}_{\mathcal{Y}}^{\mathrm{aff}})^{op}} \mathrm{Ge}_{\mathbb{G}_m}^{\text{red}}(S).$$

Informally, for  $\mathcal{Y} \in \mathrm{PreStk}$ , an object  $G \in \mathrm{Ge}_{\mathbb{G}_m}^{\text{red}}(\mathcal{Y})$  is an assignment for every  $S \in \mathrm{DGSch}_{\mathcal{Y}}^{\mathrm{aff}}$  of an object  $G_S \in \mathrm{Ge}_{\mathbb{G}_m}^{\text{red}}(S)$ , and for every  $S' \rightarrow S$  of an isomorphism

$$f^*(G_S) \simeq G_{S'}.$$

**5.5. Twistings: reformulations.** We shall see that the notion of twisting can be formulated in terms of

$$(\mathrm{Ge}_{\mathbb{G}_m}^{\text{red}})_{\mathrm{PreStk}}, (\mathrm{Ge}_{\mathbb{G}_a})_{\mathrm{PreStk}} \text{ or } (\mathrm{Ge}_{\mathbb{G}_a}^{\text{red}})_{\mathrm{PreStk}},$$

instead of  $(\mathrm{Ge}_{\mathbb{G}_m})|_{\mathrm{PreStk}}$ .

5.5.1. Consider the functors

$$\mathrm{Tw}^{/red}, \mathrm{Tw}_a, \mathrm{Tw}_a^{/red} : \mathrm{PreStk}^{op} \rightarrow \infty\text{-PicGrpd}$$

given by

$$\mathrm{Tw}^{/red}(\mathcal{Y}) := \ker \left( p_{\mathrm{dR}, \mathcal{Y}}^* : \mathrm{Ge}_{\mathbb{G}_m}^{/red}(\mathcal{Y}_{\mathrm{dR}}) \rightarrow \mathrm{Ge}_{\mathbb{G}_m}^{/red}(\mathcal{Y}) \right),$$

$$\mathrm{Tw}_a(\mathcal{Y}) := \ker \left( p_{\mathrm{dR}, \mathcal{Y}}^* : \mathrm{Ge}_{\mathbb{G}_a}(\mathcal{Y}_{\mathrm{dR}}) \rightarrow \mathrm{Ge}_{\mathbb{G}_a}(\mathcal{Y}) \right)$$

and

$$\mathrm{Tw}_a^{/red}(\mathcal{Y}) := \ker \left( p_{\mathrm{dR}, \mathcal{Y}}^* : \mathrm{Ge}_{\mathbb{G}_a}^{/red}(\mathcal{Y}_{\mathrm{dR}}) \rightarrow \mathrm{Ge}_{\mathbb{G}_a}^{/red}(\mathcal{Y}) \right).$$

We have the following diagram of functors given by the exponential map and the evident forgetful functors.

$$(5.2) \quad \begin{array}{ccc} \mathrm{Tw}_a^{/red} & \xrightarrow{\exp} & \mathrm{Tw}^{/red} \\ \downarrow & & \downarrow \\ \mathrm{Tw}_a & & \mathrm{Tw} \end{array}$$

**Proposition 5.5.2.** *The functors in (5.2) are equivalences.*

*Proof.* The functor given by the exponential map is an equivalence by Sect. 5.4.5. We construct the map inverse to the right vertical functor in (5.2) as follows. Given  $T \in \mathrm{Tw}(\mathcal{Y})$  and  $S \in \mathrm{DGSch}_{\mathcal{Y}_{\mathrm{dR}}}^{\mathrm{aff}}$  we define the trivialization of

$$G(S)|_{cl, red S}$$

to be one corresponding to the map  $cl, red S \rightarrow \mathcal{Y}$ . This defines an object of  $\mathrm{Ge}_{\mathbb{G}_m}^{/red}(\mathcal{Y}_{\mathrm{dR}})$ . The trivialization of its restriction to  $\mathcal{Y}$  follows from the construction. It is also clear that the functor thus constructed

$$\mathrm{Tw}(\mathcal{Y}) \rightarrow \mathrm{Tw}^{/red}(\mathcal{Y})$$

is the inverse to the right vertical functor in (5.2). The functor inverse to the left vertical functor is constructed in the same way.  $\square$

5.5.3. As a consequence of Proposition 5.5.2, we obtain that if  $T$  is a twisting on  $\mathcal{Y}$ , then for any affine DG scheme  $S \rightarrow \mathcal{Y}_{\mathrm{dR}}$ , the resulting  $\mathbb{G}_m$ -gerbe on  $S$  is trivialized on  $cl, red S$ . In particular, it is globally non-empty. For this reason, it does not matter which of the three versions of gerbes in Sect. 5.1.1 we choose for the definition.

Furthermore, by Proposition 5.5.2, the functor  $\mathrm{Tw}$  naturally upgrades to a functor

$$\mathrm{Tw} : \mathrm{PreStk}^{op} \rightarrow \infty\text{-PicGrpd}_k.$$

5.5.4. *Example.* We can use the natural  $k$ -module structure on  $\mathrm{Tw}$  to produce additional examples of twistings. Let  $\mathcal{L}$  be a line bundle on  $\mathcal{Y}$ , and let  $T(\mathcal{L})$  be the twisting of Sect. 5.2.3.

Now, for  $a \in k$ , the  $k$ -module structure on  $\mathrm{Tw}(\mathcal{Y})$  gives us a new twisting  $T(\mathcal{L}^{\otimes a})$ .

**5.6. Identification of the Picard groupoid of twistings.** We can use the description of twistings in terms of  $\mathbb{G}_a$ -gerbes to give a cohomological description of the groupoid of twistings.

5.6.1. *de Rham cohomology.* Let  $\mathcal{X}$  be a prestack. Recall that the coherent cohomology of  $\mathcal{X}$  is defined as

$$H(\mathcal{X}) := \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}).$$

We define the de Rham cohomology of  $\mathcal{X}$  to be the coherent cohomology of  $\mathcal{X}_{\mathrm{dR}}$ ; i.e.,

$$H_{\mathrm{dR}}(\mathcal{X}) := H(\mathcal{X}_{\mathrm{dR}}) = \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{X}_{\mathrm{dR}})}(\mathcal{O}_{\mathcal{X}_{\mathrm{dR}}}, \mathcal{O}_{\mathcal{X}_{\mathrm{dR}}}).$$

Note that since  $\mathrm{QCoh}(\mathcal{X}_{\mathrm{dR}})$  is a stable  $\infty$ -category, the  $\mathcal{M}aps$  above gives a (not necessarily connective) spectrum.

Let  $X$  be a smooth classical scheme. In this case, by Sect. 4.7, we have

$$H_{\mathrm{dR}}(X) = \mathcal{M}aps_{\mathrm{D-mod}^l(X)}(\mathcal{O}_X, \mathcal{O}_X).$$

In particular, our definition of de Rham cohomology agrees with the usual one for smooth classical schemes.

5.6.2. Consider the functor  $B^2\mathbb{G}_a$ , which represents  $\mathbb{G}_a$ -gerbes. By definition, for a prestack  $\mathcal{X}$ , we have

$$\mathcal{M}aps(\mathcal{X}, B^2\mathbb{G}_a) \simeq \Omega^{\infty-2} \mathcal{M}aps_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) \simeq \Omega^{\infty-2} H(\mathcal{X}),$$

where  $\Omega^{\infty-2}$  is the functor which assigns to a spectrum its  $(-2)$ -nd space.

Thus by Proposition 5.5.2, we obtain:

**Corollary 5.6.3.** *For a prestack  $\mathcal{X}$ , the Picard groupoid of twistings is given by*

$$Tw(\mathcal{X}) \simeq \Omega^{\infty-2} \ker(H_{\mathrm{dR}}(\mathcal{X}) \rightarrow H(\mathcal{X})).$$

5.6.4. Now, suppose that  $X$  is a smooth classical scheme. In this case, we have that

$$H_{\mathrm{dR}}(X) \simeq \Gamma(X, \Omega^\bullet)$$

where  $\Omega^\bullet$  is the complex of de Rham differentials on  $X$ . The natural map

$$H_{\mathrm{dR}}(X) \rightarrow H(X)$$

is given by global sections of the projection map  $\Omega^\bullet \rightarrow \mathcal{O}_X$ . Therefore, we have

$$Tw(X) \simeq \ker(\Omega^{\infty-2} H_{\mathrm{dR}}(X) \rightarrow \Omega^{\infty-2} H(X)) \simeq \tau^{\leq 2}(\Gamma(X, \ker(\Omega^\bullet \rightarrow \mathcal{O}_X)))[2].$$

We have that  $\tau^{\leq 2}(\ker(\Omega^\bullet \rightarrow \mathcal{O}_X))$  is given by the complex

$$\Omega^1 \rightarrow \Omega^{2,cl}$$

where  $\Omega^{2,cl}$  is the sheaf of closed 2-forms and the map is the de Rham differential. Thus, we have that the Picard groupoid of twistings on  $X$  is given by

$$Tw(X) \simeq \tau^{\leq 2}(\Gamma(X, \Omega^1 \rightarrow \Omega^{2,cl}))[2].$$

In particular, our definition of twistings agrees with the notion of TDO of [BB] for smooth classical schemes.

**5.7. Digression: invertible objects in  $\mathrm{IndCoh}$ .** Until the end of this section, we reinstate the assumption that all our DG schemes and prestacks are locally almost of finite type unless explicitly stated otherwise.

Our goal is to reformulate the functors  $\mathrm{Ge}_{\mathbb{G}_m}$  and  $Tw$  in terms of  $\mathrm{IndCoh}$  rather than  $\mathrm{QCoh}$ . For that, we will need to review the notion of an invertible object in  $\mathrm{IndCoh}$ .

5.7.1. Let  $X$  be a quasi-compact scheme locally almost of finite type. Consider the category  $\mathrm{IndCoh}(X)$ , and recall that it has a natural structure of symmetric monoidal category with respect to the  ${}^!\otimes$  operation:

$$\mathcal{F}_1 {}^!\otimes \mathcal{F}_2 := \Delta_X^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2),$$

where  $\Delta_X$  is the diagonal map  $X \rightarrow X \times X$ . The unit in  $\mathrm{IndCoh}(X)$  is  $\omega_X$ .

5.7.2. *Warning.* Unlike  $\mathrm{QCoh}(X)$ , the monoidal category  $\mathrm{IndCoh}(X)$  is, in general, not rigid in the sense of [GL:DG, Sect. 6]. (Below we shall see that  $\mathrm{IndCoh}$  is rigid if and only if  $X$  is a smooth classical scheme.)

Recall that there exists a canonical equivalence

$$\mathrm{IndCoh}(X)^\vee \simeq \mathrm{IndCoh}(X),$$

or, equivalently,

$$\mathrm{Coh}(X)^{op} = (\mathrm{IndCoh}(X)^c)^{op} \rightarrow \mathrm{IndCoh}(X)^c = \mathrm{Coh}(X),$$

given by the Serre duality functor  $\mathbb{D}_X^{Serre}$ .

This functor, however, does *not* coincide with the partially defined functor of dualization on  $\mathrm{IndCoh}(X)^c$  with respect to the monoidal structure. For example, if  $X$  is eventually coconnective in which case  $\omega_X \in \mathrm{Coh}(X)$ , the Serre dual of  $\omega_X$  is  $\Xi_X(\mathcal{O}_X)$ , while its monoidal dual is  $\omega_X$ .

5.7.3. Recall that the functor

$$\Psi_X^\vee : \mathrm{QCoh}(X) \rightarrow \mathrm{IndCoh}(X), \mathcal{F} \mapsto \mathcal{F} \otimes \omega_X$$

is naturally symmetric monoidal.

**Proposition 5.7.4.** *The functor  $\Psi_X^\vee$  defines an equivalence*

$$\mathrm{Pic}(X) \rightarrow \mathrm{Pic}(\mathrm{IndCoh}(X)),$$

where the latter denotes the category of invertible objects in  $\mathrm{IndCoh}(X)$  with respect to its symmetric monoidal structure.

In fact, we shall prove a stronger assertion:

**Proposition 5.7.5.** *The functor  $\Psi_X^\vee$  defines an equivalence between the categories of dualizable objects in  $\mathrm{QCoh}(X)$  and  $\mathrm{IndCoh}(X)$ .*

*Remark 5.7.6.* Recall (see [GL:QCoh, Lemma 4.2.2]) that an object of  $\mathrm{QCoh}(X)$  is dualizable if and only if it is perfect.

*Proof.* First, we claim that the assertion can be reduced to the case when  $X$  is eventually coconnective. Indeed, by [GL:QCoh, Lemma 4.1.5]

$$\mathrm{QCoh}(X)^{\mathrm{perf}} \simeq \lim_n \mathrm{QCoh}(\tau^{\leq n}(X))^{\mathrm{perf}}.$$

By [GL:IndCoh, Proposition 4.3.4], we also have

$$\mathrm{IndCoh}(X) \simeq \lim_n \mathrm{IndCoh}(\tau^{\leq n}(X)),$$

with the  ${}^!$ -restriction functors  $\mathrm{IndCoh}(\tau^{\leq n}(X)) \rightarrow \mathrm{IndCoh}(\tau^{\leq n'}(X))$  for  $n \geq n'$  being symmetric monoidal. Hence,

$$\mathrm{IndCoh}(X)^{\mathrm{dualizable}} \simeq \lim_n \mathrm{IndCoh}(\tau^{\leq n}(X))^{\mathrm{dualizable}}.$$



Thus, from now on we shall assume that  $X$  is eventually coconnective. In this case, by [GL:IndCoh, Corollary 8.5.1], the functor  $\Psi_X^\vee$  is fully faithful. Hence, it remains to show that it is essentially surjective as a functor

$$\mathrm{QCoh}(X)^{\mathrm{perf}} \rightarrow \mathrm{IndCoh}(X)^{\mathrm{dualizable}}.$$

Recall (see [GL:IndCoh, Proposition 8.5.7(c)]) that for  $X$  eventually coconnective,  $\omega_X \in \mathrm{IndCoh}(X)$  belongs to  $\mathrm{Coh}(X) = \mathrm{IndCoh}(X)^c$ . Hence, by [GL:DG, Lemma 5.1.1], every object  $\mathcal{F} \in \mathrm{IndCoh}(X)^{\mathrm{dualizable}}$  is compact, i.e., belongs to  $\mathrm{Coh}(X)$ .

In particular,  $\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F})$  is defined as an object of  $\mathrm{Coh}(X)$ . It is easy to see that it suffices to show that  $\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F})$ , regarded as an object of  $\mathrm{QCoh}(X)$ , is perfect.<sup>6</sup>

Let  $\mathcal{F}'$  be an arbitrary object of  $\mathrm{Coh}(X)$ . It is well-known that  $\mathcal{F}'$  is perfect if and only if for every

$$\iota_x : \mathrm{Spec}(k) \hookrightarrow X,$$

$\iota^*(\mathcal{F}') \in \mathrm{Vect}^c$ , i.e., has finitely many non-zero cohomologies.

Thus, it suffices to see that  $\mathrm{Hom}_{\mathrm{Coh}(X)}^\bullet(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), \iota_*(k))$  has finitely many cohomologies. We have:

$$\mathrm{Hom}_{\mathrm{Coh}(X)}^\bullet(\mathbb{D}_X^{\mathrm{Serre}}(\mathcal{F}), \iota_*(k)) \simeq \mathrm{Hom}_{\mathrm{Coh}(X)}^\bullet(\iota_*(k), \mathcal{F}) \simeq \iota^!(\mathcal{F}).$$

However, the functor  $\iota^! : \mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(\mathrm{Spec}(k)) = \mathrm{Vect}$  is symmetric monoidal. Therefore, since  $\mathcal{F}$  is dualizable, so is  $\iota^!(\mathcal{F})$ . Thus, the latter belongs to  $\mathrm{Vect}^c$ .  $\square$

**Corollary 5.7.7.** *Suppose the category  $\mathrm{IndCoh}(X)$  is a rigid monoidal category in the sense of [GL:DG], Sect. 6. Then  $X$  is a smooth classical scheme.*

*Proof.* Since  $\mathrm{IndCoh}(X)$  is compactly generated, if  $\mathrm{IndCoh}(X)$  rigid, then all of its compact objects are dualizable. By Proposition 5.7.5, we obtain that the functor  $\Psi_X^\vee$  induces an equivalence between  $\mathrm{QCoh}(X)^{\mathrm{perf}}$  and  $\mathrm{Coh}(X)$ . By ind-extension, we obtain that  $\Psi_X^\vee$  is an equivalence

$$\mathrm{QCoh}(X) \rightarrow \mathrm{IndCoh}(X).$$

Hence, the dual functor  $\Psi_X : \mathrm{IndCoh}(X) \rightarrow \mathrm{QCoh}(X)$  is also an equivalence. However, by [GL:IndCoh, Proposition 1.5.4], this implies that  $X$  is a smooth classical scheme.  $\square$

5.7.8. Note that by passing to the limit, Proposition 5.7.5 implies the following:

**Corollary 5.7.9.** *For any  $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$ , the functor  $\Psi_{\mathcal{Y}}^\vee$  defines an equivalence*

$$\mathrm{QCoh}(\mathcal{Y})^{\mathrm{dualizable}} \rightarrow \mathrm{IndCoh}(\mathcal{Y})^{\mathrm{dualizable}}.$$

5.7.10. As a corollary of Corollary 5.7.9 we obtain the following descent result. Let

$$\mathcal{X}^0 \rightarrow \mathcal{X}^{-1}$$

be a proper and surjective map and let  $\mathcal{X}^\bullet$  be the Čech nerve as in [GL:IndSch, Sect. 2.6.2].

**Corollary 5.7.11.** *The functor defined by the augmentation*

$$\mathrm{QCoh}(\mathcal{X}^{-1})^{\mathrm{dualizable}} \rightarrow \mathrm{Tot}(\mathrm{QCoh}(\mathcal{X}^\bullet)^{\mathrm{dualizable}})$$

*is an equivalence.*

*Proof.* By Corollary 5.7.9, the statement is equivalent to the one for  $\mathrm{IndCoh}$ . However, the latter follows from [GL:IndSch, Lemma 2.6.3].  $\square$

<sup>6</sup>The argument below is due to D. Arinkin.

### 5.8. Twisting via ind-coherent sheaves.

5.8.1. Let  $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}}^{\mathrm{IndCoh}}$  and  $(\mathrm{Ge}_{\mathbb{G}_m}^{\mathrm{IndCoh}/red})_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}}$  be the functors

$$\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}} \rightarrow \infty\text{-PicGrpd}$$

defined in a way similar to  $(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}}$  and  $(\mathrm{Ge}_{\mathbb{G}_m}^{red})_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}}$ , where instead of line bundles, we use  $\mathrm{Pic}(\mathrm{IndCoh}(-))$ .

5.8.2. The above functors give rise to the functors

$$\mathrm{Tw}_{\mathrm{PreStk}_{\mathrm{laft}}}^{\mathrm{IndCoh}} \text{ and } \mathrm{Tw}_{\mathrm{PreStk}_{\mathrm{laft}}}^{\mathrm{IndCoh}/red} : \mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}} \rightarrow \infty\text{-PicGrpd}$$

By Proposition 5.7.4, we have:

**Corollary 5.8.3.** *There exist natural equivalences*

$$(\mathrm{Ge}_{\mathbb{G}_m})_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}} \simeq (\mathrm{Ge}_{\mathbb{G}_m}^{\mathrm{IndCoh}})_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}}, \quad (\mathrm{Ge}_{\mathbb{G}_m}^{red})_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}} \simeq (\mathrm{Ge}_{\mathbb{G}_m}^{\mathrm{IndCoh}/red})_{\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}}$$

and

$$\mathrm{Tw}_{\mathrm{PreStk}_{\mathrm{laft}}}^{\mathrm{IndCoh}} \simeq \mathrm{Tw}_{\mathrm{PreStk}_{\mathrm{laft}}}^{\mathrm{IndCoh}}, \quad \mathrm{Tw}_{\mathrm{PreStk}_{\mathrm{laft}}}^{red} \simeq \mathrm{Tw}_{\mathrm{PreStk}_{\mathrm{laft}}}^{\mathrm{IndCoh}/red}$$

as functors  $(\mathrm{PreStk}_{\mathrm{laft}})^{op} \rightarrow \infty\text{-PicGrpd}$ .

### 5.9. Twistings on indschemes.

5.9.1. Let  $\mathcal{X}$  be an object of  $\mathrm{DGindSch}_{\mathrm{laft}}$ . We will show that the assertion of Corollary 5.3.3 holds for  $\mathcal{X}$ :

**Proposition 5.9.2.** *The functor*

$$\mathrm{Tw}(\mathcal{X}) \rightarrow \{\text{Central extensions of the infinitesimal groupoid of } \mathcal{X} \text{ by } \mathbb{G}_m\}$$

*is an equivalence.*

*Proof.* We shall construct an inverse functor. By Corollary 5.2.10(b), we have to construct for every  $S \in (\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}})_{/\mathcal{X}_{\mathrm{dR}}}$  a  $\mathbb{G}_m$ -gerbe on  $S$ , and a trivialization of this gerbe whenever we have a map  $S \rightarrow \mathcal{X}$ .

Consider the map

$$(5.3) \quad S \times_{\mathcal{X}_{\mathrm{dR}}} \mathcal{X} \rightarrow S.$$

Let  $S^\bullet$  be its Čech nerve. We have

$$S^\bullet \simeq S \times_{\mathcal{X}} (\mathcal{X}^\bullet / \mathcal{X}_{\mathrm{dR}}).$$

As in the proof of Proposition 3.2.3, all  $S^i$  belong to  $\mathrm{DGindSch}$ , and the morphism (5.3) satisfies the assumptions of [GL:IndSch, Sect. 2.6.2].

Given  $G_{|\mathcal{X}^\bullet / \mathcal{X}_{\mathrm{dR}}|} \in \mathrm{Ge}_{\mathbb{G}_m}(|\mathcal{X}^\bullet / \mathcal{X}_{\mathrm{dR}}|)$ , we pull it back to obtain an object  $G_{|S^\bullet|} \in \mathrm{Ge}_{\mathbb{G}_m}(|S^\bullet|)$ . Moreover, since  $G_{|\mathcal{X}^\bullet / \mathcal{X}_{\mathrm{dR}}|}$  is trivialized on  $\mathcal{X}$ , the restriction of  $G_{|S^\bullet|}$  under

$$S^0 := S \times_{\mathcal{X}_{\mathrm{dR}}} \mathcal{X} \rightarrow |S^\bullet|$$

is trivialized as well.

We need to show that  $G_{|S^\bullet|}$  canonically descends to a  $\mathbb{G}_m$ -gerbe on  $S$ . First, we claim that if a descent exists, it is canonical. I.e., that

$$\mathrm{Pic}(S) \rightarrow \mathrm{Pic}(|S^\bullet|)$$

is an equivalence. This follows from Corollary 5.7.11.

Thus, it remains to prove the existence. By assumption, the descent is possible over  $^{cl, red}S$ . By convergence, we can assume that  $S$  is eventually coconnective. Thus, by induction, we can assume that the descent takes place over a closed subscheme  $S_1 \subset S$  given by a square-zero ideal,  $\mathcal{J} \in \mathrm{QCoh}(S_1)$ . Since  $S_1$  is affine, we can assume that the descended gerbe is trivial.

The datum of a  $\mathbb{G}_m$ -gerbe on  $|S^\bullet|$ , trivialized over  $S_1 \times_S |S^\bullet|$ , is equivalent to that of a map

$$\mathcal{O}_{S_1 \times_S S^\bullet} \rightarrow \mathcal{J}|_{S_1 \times_S S^\bullet}[2] \in \mathrm{Tot} \left( \mathrm{QCoh}(S_1 \times_S S^\bullet) \right).$$

By eventual connectivity and coherence, the latter is the same as a morphism

$$\omega_{S_1 \times_S S^\bullet} \rightarrow \Psi_{S_1 \times_S S^\bullet}^\vee(\mathcal{J}|_{S_1 \times_S S^\bullet})[2] \in \mathrm{Tot} \left( \mathrm{IndCoh}(S_1 \times_S S^\bullet) \right).$$

Now, by [GL:IndSch, Lemma 2.6.3], the latter canonically comes from a map

$$\omega_{S_1} \rightarrow \Psi_{S_1}^\vee(\mathcal{J})[2] \in \mathrm{IndCoh}(S_1),$$

which is the same as a morphism

$$\mathcal{O}_{S_1} \rightarrow \mathcal{J}[2] \in \mathrm{QCoh}(S),$$

which is what we had to show.  $\square$

## 6. TWISTED CRYSTALS

**6.1. Twisted left crystals.** In this subsection we do not assume that our DG schemes and prestacks are locally almost of finite type.

6.1.1. Let  $\mathcal{Y}$  be a prestack. Consider the category  $\mathrm{PreStk}/\mathcal{Y}$ , and the functor

$$\mathrm{QCoh}_{\mathrm{DGSch}/\mathcal{Y}}^{\mathrm{aff}} : (\mathrm{DGSch}/\mathcal{Y})^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}.$$

The group stack  $B\mathbb{G}_m$  acts on  $\mathrm{QCoh}$  via tensoring by line bundles.

Let  $G$  be a  $\mathbb{G}_m$ -gerbe on  $\mathcal{Y}$ .  $G$  gives a twist of the functor  $\mathrm{QCoh}_{\mathrm{DGSch}/\mathcal{Y}}^{\mathrm{aff}}$  via the action of  $B\mathbb{G}_m$  on  $\mathrm{QCoh}$ . This gives a functor

$$\mathrm{QCoh}_{\mathrm{DGSch}/\mathcal{Y}}^G : (\mathrm{DGSch}/\mathcal{Y})^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}.$$

6.1.2. In particular, if  $T$  is a twisting on  $\mathcal{Y}$ , we obtain a functor

$$\mathrm{QCoh}_{\mathrm{DGSch}/\mathcal{Y}_{\mathrm{dR}}}^T : (\mathrm{DGSch}/\mathcal{Y}_{\mathrm{dR}})^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}.$$

Let  $\mathrm{QCoh}_{\mathrm{DGSch}/\mathcal{Y}}^T$  be its restriction along the map

$$(\mathrm{DGSch}/\mathcal{Y})^{\mathrm{op}} \rightarrow (\mathrm{DGSch}/\mathcal{Y}_{\mathrm{dR}})^{\mathrm{op}}.$$

We then have that  $\mathrm{QCoh}_{\mathrm{DGSch}/\mathcal{Y}}^T$  is canonically isomorphic to  $\mathrm{QCoh}_{\mathrm{DGSch}/\mathcal{Y}}^{\mathrm{aff}}$ .

6.1.3. More generally, we can consider the functor

$$\mathrm{QCoh}_{\mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}}}^T : (\mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}},$$

which is the right Kan extension of  $\mathrm{QCoh}_{\mathrm{DGSch}/\mathcal{Y}_{\mathrm{dR}}}^T$  along

$$(\mathrm{QCoh}_{\mathrm{DGSch}/\mathcal{Y}_{\mathrm{dR}}}^T)^{op} \hookrightarrow (\mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}})^{op}.$$

The restriction  $\mathrm{QCoh}_{\mathrm{PreStk}/\mathcal{Y}}^T$  of  $\mathrm{QCoh}_{\mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}}}^T$  along

$$\mathrm{PreStk}/\mathcal{Y} \rightarrow \mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}}$$

is canonically isomorphic to  $\mathrm{QCoh}_{\mathrm{PreStk}/\mathcal{Y}}$ .

6.1.4. For a twisting  $T$  on a prestack  $\mathcal{Y}$ , the category of  $T$ -twisted left crystals on  $\mathcal{Y}$  is defined as

$$\mathrm{Crys}^{T,l}(\mathcal{Y}) := \mathrm{QCoh}^T(\mathcal{Y}_{\mathrm{dR}}).$$

Explicitly, we have

$$\mathrm{Crys}^{T,l}(\mathcal{Y}) = \lim_{S \in (\mathrm{DGSch}/\mathcal{Y}_{\mathrm{dR}})^{op}} \mathrm{QCoh}^T(S).$$

6.1.5. More generally, we define the functor

$$\mathrm{Crys}_{\mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}}}^{T,l} : (\mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}},$$

as the composite  $\mathrm{QCoh}_{\mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}}}^T \circ \mathrm{dR}$ . The analogue of Corollary 2.1.4 holds for this functor.

6.1.6. We have a canonical natural transformation

$$\mathbf{oblv}^{T,l} : \mathrm{Crys}_{\mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}}}^{T,l} \rightarrow \mathrm{QCoh}_{\mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}}}^T.$$

For an individual  $\mathcal{Y}' \in \mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}}$ , we denote the resulting functor

$$\mathrm{Crys}^{T,l}(\mathcal{Y}') \rightarrow \mathrm{QCoh}^T(\mathcal{Y}')$$

by  $\mathbf{oblv}^{T,l}(\mathcal{Y}')$ .

6.1.7. Let  $\mathrm{Crys}_{\mathrm{PreStk}/\mathcal{Y}}^{T,l}$  denote the restriction of  $\mathrm{Crys}_{\mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}}}^{T,l}$  along  $\mathrm{PreStk}/\mathcal{Y} \rightarrow \mathrm{PreStk}/\mathcal{Y}_{\mathrm{dR}}$ .

By a slight abuse of notation we shall use the same symbol  $\mathbf{oblv}^{T,l}$  to denote the resulting natural transformation

$$\mathrm{Crys}_{\mathrm{PreStk}/\mathcal{Y}}^{T,l} \rightarrow \mathrm{QCoh}_{\mathrm{PreStk}/\mathcal{Y}}.$$

**6.2. Twisted right crystals.** At this point, we reinstate the assumption that all DG schemes and prestacks are locally almost of finite type for the rest of the paper.

6.2.1. Let  $\mathcal{Y}$  be an object of  $\mathrm{PreStk}_{\mathrm{lft}}$ , and let  $G$  be a  $\mathbb{G}_m$ -gerbe on  $\mathcal{Y}$ .

The action of  $\mathrm{QCoh}_{\mathrm{PreStk}_{\mathrm{lft}}}$  on  $\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{lft}}}$  allows to define the functor

$$\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{lft}})/\mathcal{Y}}^G : ((\mathrm{PreStk}_{\mathrm{lft}})/\mathcal{Y})^{op} \rightarrow \mathrm{DGCat}_{\mathrm{cont}},$$

with properties analogous to those of

$$\mathrm{IndCoh}_{(\mathrm{PreStk}_{\mathrm{lft}})/\mathcal{Y}} := \mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{lft}}} |_{(\mathrm{PreStk}_{\mathrm{lft}})/\mathcal{Y}}.$$

6.2.2. In particular, for  $T \in \text{Tw}(\mathcal{Y})$ , we have the functor

$$\text{Crys}_{(\text{PreStk}_{\text{laft}})/\mathcal{Y}_{\text{dR}}}^{T,r} : ((\text{PreStk}_{\text{laft}})/\mathcal{Y}_{\text{dR}})^{op} \rightarrow \text{DGCat}_{\text{cont}},$$

and the natural transformations  $\mathbf{oblv}^{T,l}$

$$\text{Crys}_{(\text{PreStk}_{\text{laft}})/\mathcal{Y}_{\text{dR}}}^{T,r} \rightarrow \text{IndCoh}_{(\text{PreStk}_{\text{laft}})/\mathcal{Y}_{\text{dR}}}^T \quad \text{and} \quad \text{Crys}_{(\text{PreStk}_{\text{laft}})/\mathcal{Y}}^{T,r} \rightarrow \text{IndCoh}_{(\text{PreStk}_{\text{laft}})/\mathcal{Y}}.$$

The analogues of Corollaries 2.3.7, 2.3.9, Corollary 2.3.12, and Lemma 2.3.11 hold for  $\mathcal{Y}' \in (\text{PreStk}_{\text{laft}})/\mathcal{Y}_{\text{dR}}$ , with the same proofs.

**6.3. Properties of twisted crystals.** As was mentioned above, all DG schemes and prestacks are assumed locally almost of finite type.

Let  $\mathcal{Y}$  be a fixed object of  $\text{PreStk}_{\text{laft}}$ , and  $T \in \text{Tw}(\mathcal{Y})$ .

6.3.1. The analogues of Corollaries 2.2.2, 2.2.4, 2.1.7 and Lemma 2.2.6 hold for twisted left crystals, with the same proofs.

Furthermore, Kashiwara's lemma holds for both left and right twisted crystals, also with the same proof.

Finally, note that there exists a canonical natural transformation

$$(6.1) \quad \Upsilon : \text{Crys}_{(\text{PreStk}_{\text{laft}})/\mathcal{Y}_{\text{dR}}}^{T,l} \rightarrow \text{Crys}_{(\text{PreStk}_{\text{laft}})/\mathcal{Y}_{\text{dR}}}^{T,r}.$$

**Proposition 6.3.2.** *The natural transformation (6.1) is an equivalence.*

*Proof.* The argument is identical to that of Proposition 2.4.4. Note we do not need the smooth classical scheme  $Z$  to map to  $\mathcal{Y}$ . Rather, we are using that Propositions 7.1.3, 7.4.5 and 8.4.4 of [GL:IndSch] hold in the situation when  $\text{QCoh}(-)$  and  $\text{IndCoh}(-)$  are replaced by their versions twisted by a  $\mathbb{G}_m$ -gerbe.<sup>7</sup>

□

6.3.3. Hence, for  $\mathcal{Y}' \in (\text{PreStk}_{\text{laft}})/\mathcal{Y}_{\text{dR}}$  we can regard crystals on  $\mathcal{Y}'$  as a single category,  $\text{Crys}^T(\mathcal{Y}')$ , endowed with two forgetful functors

$$(6.2) \quad \begin{array}{ccc} & \text{Crys}^T(\mathcal{Y}') & \\ \mathbf{oblv}_{\mathcal{Y}'}^{T,l} \swarrow & & \searrow \mathbf{oblv}_{\mathcal{Y}'}^{T,r} \\ \text{QCoh}^T(\mathcal{Y}') & \xrightarrow{\Psi_{\mathcal{Y}'}^{\vee}} & \text{IndCoh}^T(\mathcal{Y}') \end{array}$$

For  $\mathcal{Y}' \in (\text{PreStk}_{\text{laft}})/\mathcal{Y}$ , the above forgetful functors map to non-twisted sheaves:

$$(6.3) \quad \begin{array}{ccc} & \text{Crys}^T(\mathcal{Y}') & \\ \mathbf{oblv}_{\mathcal{Y}'}^{T,l} \swarrow & & \searrow \mathbf{oblv}_{\mathcal{Y}'}^{T,r} \\ \text{QCoh}(\mathcal{Y}') & \xrightarrow{\Psi_{\mathcal{Y}'}^{\vee}} & \text{IndCoh}(\mathcal{Y}') \end{array}$$

---

<sup>7</sup>The latter statement is particularly easy in our case, as all the gerbes involved are non-canonically trivial.

6.3.4. Let  $\mathcal{X} \in (\mathrm{DGindSch}_{\mathrm{laft}})_{/\mathcal{Y}}$ . The analogue of Proposition 3.2.3 holds with no change. In particular, we obtain a functor

$$\mathbf{ind}_{\mathcal{X}}^{T,r} : \mathrm{IndCoh}(\mathcal{X}) \rightarrow \mathrm{Crys}^{T,r}(\mathcal{X})$$

left adjoint to  $\mathbf{oblv}_{\mathcal{X}}^{T,r}$ .

Similarly, the analogue of Proposition 3.3.2 holds in the present context as well.

The following observation will be useful in the sequel:

**Lemma 6.3.5.** *Let  $X$  be an affine DG scheme (or an ind-affine DG indscheme). Then the monad*

$$\mathbf{oblv}_X^{T,r} \circ \mathbf{ind}_X^{T,r}$$

*considered as a plain functor  $\mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(X)$  is non-canonically isomorphic to  $\mathbf{oblv}_X^r \circ \mathbf{ind}_X^r$ .*

*Proof.* First, we observe that the analogue of Proposition 3.4.3 holds, in the sense that the object of  $\mathrm{IndCoh}(X \times X)$  that defines the functor  $\mathbf{oblv}_X^{T,r} \circ \mathbf{ind}_X^{T,r}$  identifies with

$$(\widehat{\Delta}_X)_*^{\mathrm{IndCoh}} \left( \mathcal{L} \otimes (\omega_X \times_{X_{\mathrm{dR}}} X) \right),$$

where  $\mathcal{L}$  is the line bundle on  $X \times_{X_{\mathrm{dR}}} X$  corresponding to  $T$ , as in Sect. 5.3.1.

By construction,  $\mathcal{L}$  is trivial when restricted to  $X \hookrightarrow X \times_{X_{\mathrm{dR}}} X$ . Now, since  $X$  is affine, this implies that  $\mathcal{L}$  can be trivialized on all of  $X \times_{X_{\mathrm{dR}}} X$ . □

6.3.6. In general, results about usual crystals do not automatically hold for twisted crystals for the following reason: some of our proofs relied on the possibility of embedding a given affine DG scheme  $X$  into a smooth classical scheme  $Z$ . The problem is that we might not be able to find such a  $Z$  which also maps to  $\mathcal{Y}$ .

However, there is a large family of examples (which covers all the cases that have appeared in applications so far), where the extension of the results is automatic: namely, when  $T$  is such that its restriction to any  $S \in (\mathrm{DGSch}_{\mathrm{laft}}^{\mathrm{aff}})_{/\mathcal{Y}}$  is locally trivial in the Zariski or étale topology. This is the case for twistings of the form  $\mathcal{L}^{\otimes a}$  for  $\mathcal{L} \in \mathrm{Pic}(\mathcal{Y})$  and  $a \in k$ , and tensor products thereof.

**6.4. t-structures on twisted crystals.** We remind the reader that all DG schemes and prestacks are assumed locally almost of finite type, unless explicitly stated otherwise.

As in the previous subsection, we let  $\mathcal{Y}$  be a fixed object of  $\mathrm{PreStk}_{\mathrm{laft}}$ , and  $T \in \mathrm{Tw}(\mathcal{Y})$ .

6.4.1. If  $X$  is a DG scheme and  $G$  is a  $\mathbb{G}_m$ -gerbe on it, the twisted categories  $\mathrm{QCoh}^G(X)$  and  $\mathrm{IndCoh}^G(X)$  have natural t-structures with properties analogous to those of their usual counterparts  $\mathrm{QCoh}(X)$  and  $\mathrm{IndCoh}(X)$ .

This allows to define the “left” t-structure on  $\mathrm{Crys}^{T,l}(\mathcal{Y}')$  for any  $\mathcal{Y}' \in (\mathrm{PreStk}_{\mathrm{laft}})_{/\mathcal{Y}_{\mathrm{dR}}}$ . (This t-structure can be defined without the locally almost of finite type assumption on either  $\mathcal{Y}$  or  $\mathcal{Y}'$ .)

Similarly, if  $\mathcal{Y}' = X$  is a DG scheme or an Artin stack, we can introduce the “right” t-structure on  $\mathrm{Crys}^{T,r}(\mathcal{Y}')$ .

6.4.2. We observe that Proposition 4.2.7 renders to the twisted context with no change. We now claim:

**Proposition 6.4.3.**

- (a) *The functor  $\mathbf{ind}_X^{T,r}$  is t-exact.*
- (b) *For a quasi-compact scheme  $X$ , the functor  $\mathbf{oblv}_X^{T,r}$  is of bounded cohomological amplitude.*

*Proof.* The functor  $\mathbf{ind}_X^{T,r}$  is right t-exact, since its right adjoint  $\mathbf{oblv}_X^{T,r}$  is left t-exact. By the definition of the “right” t-structure, the left t-exactness of  $\mathbf{ind}_X^{T,r}$  is equivalent to the same property of the composition  $\mathbf{oblv}_X^{T,r} \circ \mathbf{ind}_X^{T,r}$ . Now, the assertion follows from Lemma 6.3.5 and the fact that the analogous assertion holds in the non-twisted case.

Since  $\mathrm{IndCoh}^T(X)^{\leq 0}$  generates  $\mathrm{Crys}^{T,l}(X)^{\leq 0}$  via the functor  $\mathbf{ind}_X^{T,r}$ , in order to show that the cohomological amplitude of  $\mathbf{oblv}_X^{T,r}$  is bounded from above, it suffices to show the same for  $\mathbf{oblv}_X^{T,r} \circ \mathbf{ind}_X^{T,r}$ . Again, the assertion follows from Lemma 6.3.5.  $\square$

Our current goal is to show:

**Proposition 6.4.4.**

- (a) *The “right” t-structure on  $\mathrm{Crys}^{T,r}(X)$  is left-complete.*
- (b) *For  $X$  affine, the natural functor  $D(\mathrm{Crys}^{T,r}(X)^\heartsuit) \rightarrow \mathrm{Crys}^{T,r}(X)$  is an equivalence, where the heart is taken with respect to the “right” t-structure.*
- (c) *The “left” and “right” t-structures on  $\mathrm{Crys}^T(X)$  are at a finite amplitude from one another.*
- (d) *The functor  $\mathbf{oblv}_X^{T,l} : \mathrm{Crys}^T(X) \rightarrow \mathrm{QCoh}(X)$  is of bounded cohomological amplitude (by point (c) this statement does not depend on which of the two t-structures we consider on  $\mathrm{Crys}^T(X)$ ).*

Before we begin the proof we need to revisit the t-structure on the category  $\mathrm{IndCoh}$  of a formal completion.

6.4.5. *t-structure on  $\mathrm{IndCoh}$  of ind schemes.* Let  $X$  be a closed subset of a quasi-compact DG scheme  $Z$ . Let  $Y$  denote the formal completion of  $Z$  along  $X$ . Recall from [GL:IndSch, Sect. 2.4.9 and Lemma 7.4.8] that there is a natural t-structure on  $\mathrm{IndCoh}(Y)$  such that the functor

$$(\widehat{i})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Y) \simeq \mathrm{IndCoh}(Z)_X \hookrightarrow \mathrm{IndCoh}(Z)$$

is exact.

This t-structure is characterized as follows:  $\mathcal{F} \in \mathrm{IndCoh}(Y)^{\geq 0}$  if and only if for every closed embedding  $i_1 : X_1 \rightarrow Y$ , we have  $i_1^!(\mathcal{F}) \in \mathrm{IndCoh}(X_1)^{\geq 0}$ .

Equivalently, the category  $\mathrm{IndCoh}(Y)^{\leq 0}$  is generated by the essential images of

$$(i_1)_*^{\mathrm{IndCoh}}(\mathrm{IndCoh}(X_1)^{\leq 0})$$

for all closed embeddings  $i_1$  as above.

Hence, if  $G$  is a  $\mathbb{G}_m$ -gerbe on  $Y$ , the category  $\mathrm{IndCoh}^G(Y)$  also acquires a natural t-structure with analogous properties.

6.4.6. The following is an analogue of Proposition 4.2.5(b):

**Proposition 6.4.7.** *Let  $X$  be a closed subset of a smooth scheme  $Z$ , and let  $Y$  be the formal completion of  $Z$  along  $X$ . Then, the functor*

$$\mathbf{oblv}_Y^{T,r} : \mathrm{Crys}^{T,r}(Y) \rightarrow \mathrm{IndCoh}^T(Y)$$

*is t-exact.*

*Proof.* As in the proof of Proposition 6.4.3, it suffices to show that the functor

$$\mathbf{oblv}_Y^{T,r} \circ \mathbf{ind}_Y^{T,r} : \mathrm{IndCoh}^T(Y) \rightarrow \mathrm{IndCoh}^T(Y)$$

is t-exact. However, as in the proof of Proposition 6.4.3, this functor is non-canonically isomorphic to the non-twisted version:  $\mathbf{oblv}_Y^r \circ \mathbf{ind}_Y^r$ , and the latter is known to be exact by Proposition 4.2.5(b).  $\square$

6.4.8. *Proof of Proposition 6.4.4.* We can assume that  $X$  is affine and embed it into a smooth classical scheme  $Z$ . Let  $Y$  denote the formal neighborhood of  $X$  in  $Z$ . By definition,  $T$  defines a  $\mathbb{G}_m$ -gerbe  $G$  on  $Y$ . Let  $i$  denote the closed embedding  $X \hookrightarrow Y$ .

In order to prove point (a), it suffices to exhibit a collection of objects  $\mathcal{P}_\alpha \in \mathrm{Crys}^T(X)$  that generate  $\mathrm{Crys}^T(X)$  and that are of bounded Ext dimension, i.e., if for each  $\alpha$  there exists an integer  $k_\alpha$  such that

$$\mathrm{Hom}_{\mathrm{Crys}^T(X)}(\mathcal{P}_\alpha, \mathcal{M}) = 0 \text{ if } \mathcal{M} \in \mathrm{Crys}^T(X)^{<-k_\alpha}.$$

We realize  $\mathrm{Crys}^T(X)$  as  $\mathrm{Crys}^T(Y)$ . By Proposition 4.2.7 (applied to the twisted case), the corresponding t-structure on  $\mathrm{Crys}^{T,r}(Y)$  is characterized by the property that

$$\mathcal{M} \in \mathrm{Crys}^{T,r}(Y)^{\geq 0} \Leftrightarrow \mathbf{oblv}_Y^{T,r}(\mathcal{M}) \in \mathrm{IndCoh}^T(\mathcal{Y})^{\geq 0},$$

where we consider the t-structure on  $\mathrm{IndCoh}^T(\mathcal{Y})$  introduced in Sect. 6.4.5.

We take  $\mathcal{P}_\alpha$  to be of the form  $\mathbf{ind}_Y^{T,r}(\mathcal{F})$  for  $\mathcal{F} \in \mathrm{Coh}^T(Y)^\heartsuit$ . To prove the required vanishing of Exts we need to show that for  $\mathcal{M} \in \mathrm{Crys}^T(Y)^{\leq 0}$ ,

$$\mathrm{Hom}_{\mathrm{IndCoh}^T(Y)}(\mathcal{F}, \mathbf{oblv}_Y^{T,r}(\mathcal{M})) = 0.$$

By Proposition 6.4.7, we have  $\mathbf{oblv}_Y^{T,r}(\mathcal{M}) \in \mathrm{IndCoh}^T(Y)^{\leq 0}$ . Now, the assertion follows by adjunction from the fact that the category  $\mathrm{IndCoh}^T(Y)$  has a finite cohomological dimension: indeed, the category in question is non-canonically equivalent to  $\mathrm{IndCoh}(Y)$ , and the cohomological dimension of the latter is bounded by that of  $Z$ .

Let us now prove point (b). As in the proof of Corollary 4.5.8, given what we have shown in point (a), we only have to verify that for  $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Crys}^T(Y)^\heartsuit$  and any  $k$ , the map

$$\mathrm{Ext}_{\mathrm{Crys}^T(Y)^\heartsuit}^k(\mathcal{M}_1, \mathcal{M}_2) \rightarrow \mathrm{Hom}_{\mathrm{Crys}^T(Y)}(\mathcal{M}_1, \mathcal{M}_2[k])$$

is an isomorphism. For that it suffices to show that the category  $\mathrm{Crys}^T(Y)^\heartsuit$  contains a pro-projective generator of  $\mathrm{Crys}^T(Y)$ , i.e., that there exists a filtered inverse family with surjective maps  $\mathcal{P}_\alpha \in \mathrm{Crys}^T(Y)^\heartsuit$ , such that the functor

$$\mathrm{colim}_\alpha \mathrm{Maps}_{\mathrm{Crys}^T(Y)}(\mathcal{P}_\alpha, -)$$

is t-exact and conservative on  $\mathrm{Crys}^T(Y)$ . We take  $\mathcal{P}_\alpha$  to be

$$\mathbf{ind}_Y^{T,r}(\mathcal{O}_{X_n}),$$



where  $X_n$  is the  $n$ -th infinitesimal neighborhood of  ${}^{cl,red}X$  in  $Z$ .

To prove point (c), we can replace  $X$  by  $Y$ , and it suffices to show that the discrepancy between the two t-structures on  $\text{Crys}^T(Y)$  is finite. By Proposition 4.1.3 (applied in the twisted case) and Proposition 6.4.7, it suffices to show that the functor

$$\Psi_Y : \text{QCoh}^T(Y) \rightarrow \text{IndCoh}^T(Y)$$

is of bounded cohomological amplitude. This is equivalent to the corresponding fact for

$$\Psi_Y : \text{QCoh}(Y) \rightarrow \text{IndCoh}(Y),$$

which in turn follows from the corresponding fact for  $Z$ .

Point (d) follows from the fact that the functor

$$'i^* : \text{QCoh}^T(Y) \rightarrow \text{QCoh}^T(X)$$

is of bounded amplitude, which is again equivalent to the corresponding fact for

$$'i^* : \text{QCoh}(Y) \rightarrow \text{QCoh}(X),$$

which in turn follows from the corresponding fact for  $Z$ . □

6.4.9. The rest of the results concerning the behavior of the two t-structures, established in Sects. 4.4, 4.5 and 4.6 for untwisted crystals, render automatically to the twisted situation.

## 6.5. Other results.

6.5.1. *Twisted crystals and twisted D-modules.* Let  $X$  be a smooth classical scheme. We have seen in Sect. 5.6.4 that the Picard category of twistings on  $X$  is equivalent to that of TDO's on  $X$ .

As in Sect. 4.7, one shows that given a twisting  $T$ , and the corresponding TDO, denoted  $\text{Diff}^T(X)$ , there exists a canonical equivalence

$$\text{Crys}^{T,l}(X) \simeq \text{D-mod}^{T,l}(X),$$

which commutes with the forgetful functors to  $\text{QCoh}(X)$ , and similarly for twisted right crystals.

6.5.2. Now, let  $X$  be a quasi-compact DG scheme and  $T$  a locally trivial twisting on  $X$  (see Sect. 6.3.6). We then automatically obtain the following proposition from the analogous statement in the non-twisted case.

**Proposition 6.5.3.** *Under the above circumstances:*

- (a) *The abelian category  $\text{Crys}^{T,r}(X)^\heartsuit$  is Noetherian.*
- (b)  *$\text{Crys}^{T,r}(X)$  has finite cohomological dimension.*

*Remark 6.5.4.* We certainly expect the statement of Proposition 6.5.3 to hold for any (not necessarily locally trivial) twisting, but we do not currently have a proof.

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